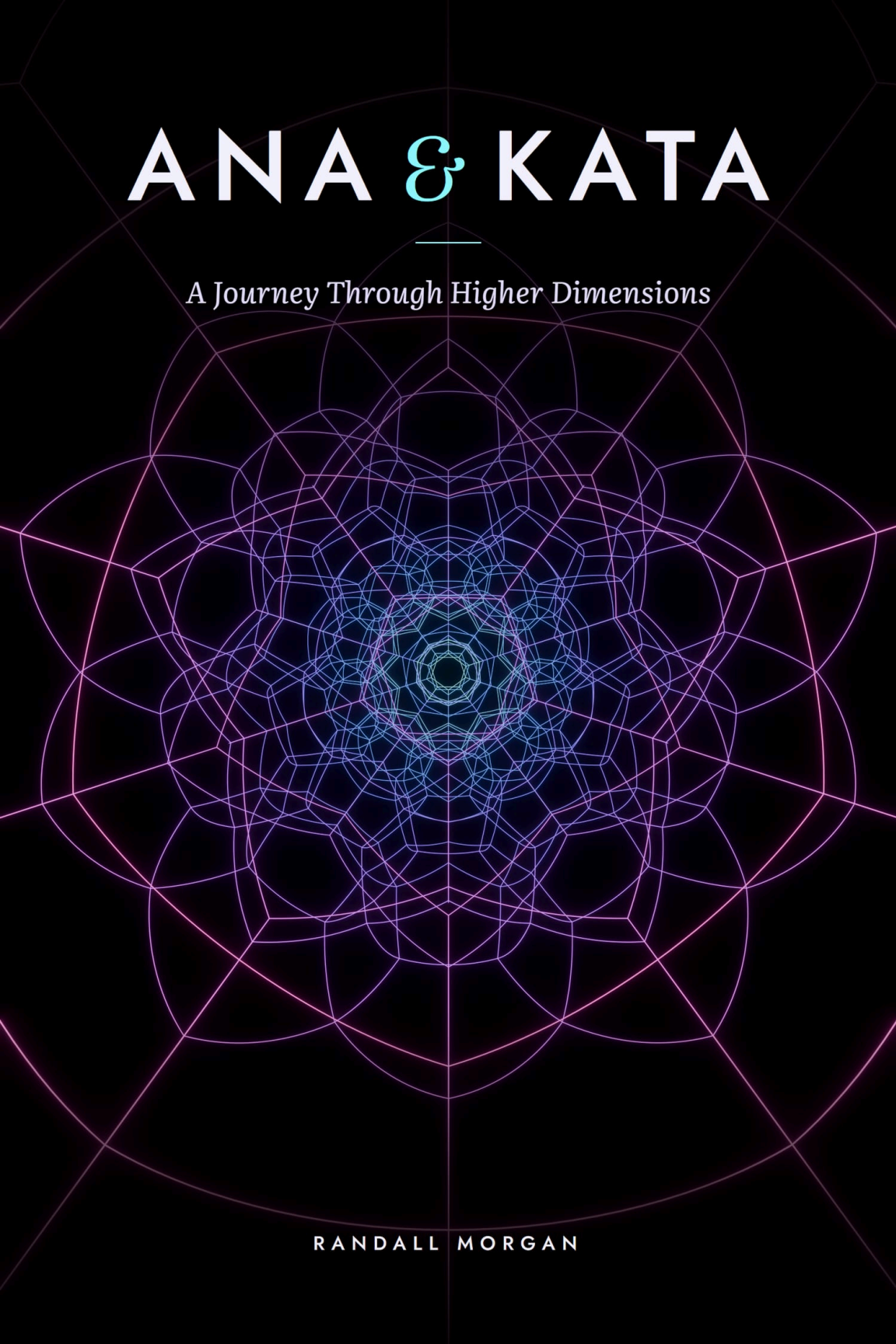
ANA & KATA

A Journey Through Higher Dimensions



RANDALL MORGAN

Ana & Kata

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Pardesco LLC

Ana & Kata: A Journey Through Higher Dimensions (free sampler)

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Every polytope image in this book was rendered by the author in Blender with the 4D Polytope Engine. Explore the shapes yourself, free, at 4d.pardesco.com.

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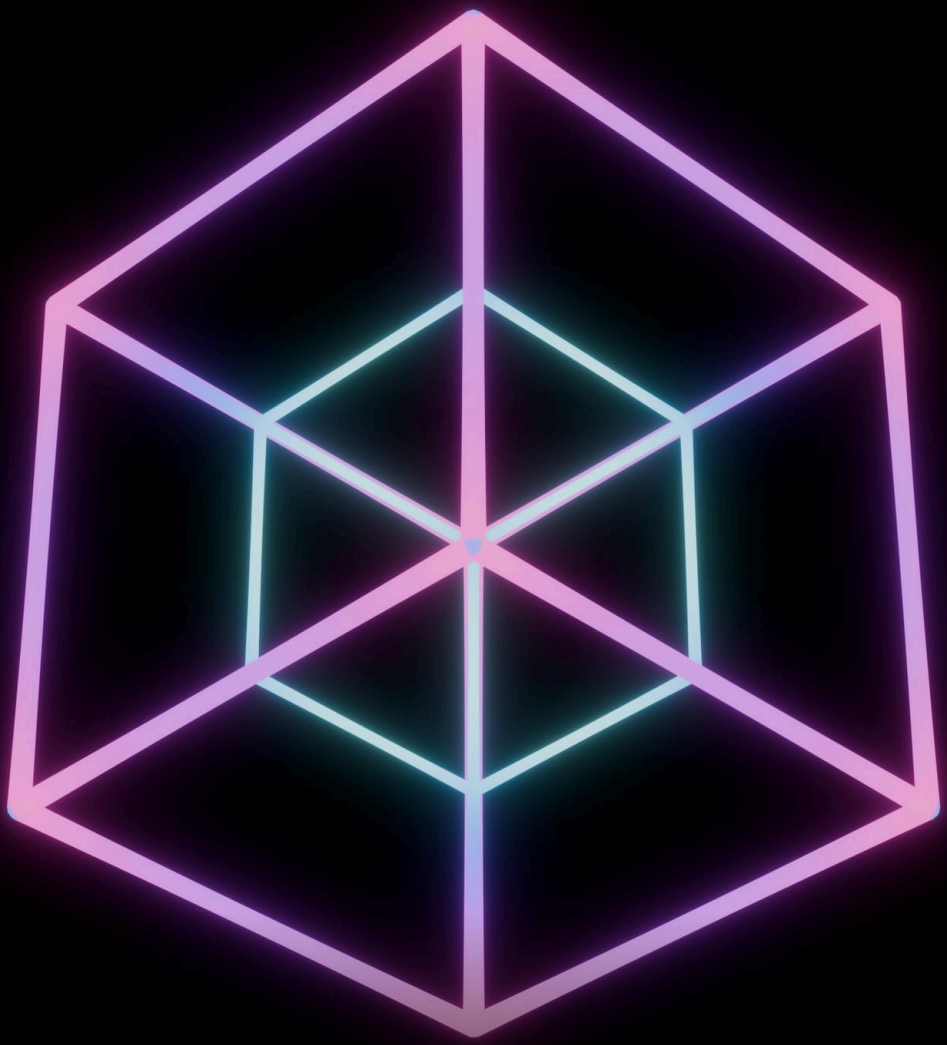
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A tesseract, the four-dimensional cube, in perspective: eight cubes, seen here as a cube within a cube.

Introduction: The Glove Problem

A pair of gloves

Lay a pair of gloves on the table, palms up. Pick up the left one and slide it across to the right. The thumb is on the wrong side. Spin it around. Now the fingers point the wrong way. Flip it upside down and try again. However you turn it, you still have a left glove.

Use matching gloves, so differences in size or wear don't distract from the puzzle. The distance between any two fingertips on one glove equals the distance between the corresponding fingertips on the other. All the lengths and angles agree. Yet no amount of sliding or turning will make the left glove occupy exactly the space the right glove occupied. There is no faulty measurement to find. The gloves are mirror images.

Mathematicians call a shape like this chiral: it has a handedness that sliding and rotation in ordinary space cannot change. The property is called chirality. Your hands are a pair of examples you can keep in front of you while reading.

A thin, flexible glove does offer an escape. Turn it inside out and you can wear it on the other hand. Try it if you have a suitable glove. The thumb ends up where you need it, with the inside on the outside. That move bends the material, though, and bending is outside the rules of this puzzle. A glove cast in metal has no inside-out move. From here on, think of the gloves as rigid.

In Leipzig in 1827, August Ferdinand Möbius took up the geometrical problem in paragraph 140 of *Der barycentrische Calcul*, or *The Barycentric Calculus*. He was an astronomer at the university observatory, whose director he would become in 1848. His

biographer Baltzer said he "worked without hurrying," holding back work until he judged it mature. He also struggled to attract paying students and advertised some courses as free.

His argument starts with a line. Suppose two arrangements of points on it are mirror images that cannot be matched by sliding along the line. Give one arrangement a half-turn through a plane, the flat space containing the line, and it can match the other. The points leave the line during the turn and return to it in the reversed arrangement.

Move up a dimension. A scalene triangle, whose sides all have different lengths, cannot be made to match its mirror image by sliding or turning within its plane. Lift it out of that flat space and give it a half-turn through ordinary space. It can now fit exactly where its counterpart was.

For mirror-image solids, Möbius reasoned, the same move would need a space of four dimensions. He had worked out the turn that would solve the glove problem, then ruled it out. In translation: "But since such a space cannot be thought, the coincidence in this case is also impossible."

Coincidence here means that the two shapes occupy exactly the same space. Within one paragraph, Möbius had described the required motion and rejected the space in which it could happen.

Lifting it out of the page

Cut a left-hand outline from paper and lay it on the table. For this experiment, treat the tabletop as the whole space available to the cut-out. Push it anywhere you please and turn it while keeping it flat. It stays a left-hand outline.

To make it a right-hand outline, lift it off the table, turn it over, and lay it down again. During that half-turn it leaves the flat space to which you had confined it. It returns with its handedness reversed. You haven't stretched the paper or changed the distances between its fingertips.

Möbius's argument asks you to carry that move one dimension higher. A rigid left glove would swing out of our three-dimensional space through a fourth spatial direction. After a half-turn, it would return as a right glove. It would keep its shape throughout the motion, with the inside still inside. The paper experiment gives you a way to follow the reasoning, even if you cannot picture the glove halfway through its turn.

The difficulty is pointing to the direction in which to start. Up from the tabletop was available to the paper hand. The glove would need a direction at right angles to all three directions of our space. Pointing upwards takes it along one of the directions we already have.

The calculation is easier to follow. Coordinates are numbers that specify a position. Call the ordinary coordinates x, y and z, and the fourth coordinate w. A half-turn in the plane containing the x and w directions changes (x, y, z, w) into (−x, y, z, −w). For a solid in our space, w = 0. After the turn, w is zero again, every x is reversed, and y and z are unchanged. The solid is back in our space as its mirror image.

This is established mathematics: an exact description of a motion in four-dimensional space. It gives us no physical means of lifting a glove into that space.

TRY IT Open the free viewer at 4d.pardesco.com/viewer and choose the tesseract under Select Polytope. Switch on XW under Auto planes and nothing else. Watch the small inner cube swell until it becomes the outer cube while the outer one shrinks inside. The object hasn't changed at all; only its shadow has.

Thirty-one years later, in 1858, Möbius described the one-sided band now called the Möbius strip. An asymmetric flat figure sliding once around that loop returns to its starting place as its own mirror image: a clockwise arrow comes back counterclockwise. Here the surface itself twists, allowing a reversal that movement within an

ordinary plane cannot produce. Johann Benedict Listing found the strip independently that same year and is usually given priority; Möbius's memoir on one-sided surfaces was found only after his death.

Who else noticed

In 1768 Immanuel Kant used right and left hands to ask what space itself must be like. He called them incongruent counterparts: bodies alike in their measurements that cannot fit within the same boundaries. The gloves had already become a philosophical problem before Möbius worked out his turn.

Kant returned to them in his *Prolegomena to Any Future Metaphysics* in 1783. In Paul Carus's translation, he writes that "the glove of one hand cannot be used for the other." Keep our rigid-glove rule in mind as you read that sentence.

By 1783 Kant used the puzzle to argue that space is a form of our perception, a speculative philosophical proposal that geometry alone cannot establish; Chapter 4 takes up his argument.

Ludwig Wittgenstein gave the geometrical answer in his *Tractatus Logico-Philosophicus*. In C. K. Ogden's 1922 translation: "A right-hand glove could be put on a left hand if it could be turned round in four-dimensional space." He allowed the extra room that Möbius had rejected.

In H. G. Wells's "The Plattner Story," published in April 1896, a fictional schoolmaster gets that room by accident. Gottfried Plattner teaches modern languages and also chemistry, a subject he knows almost nothing about. A pupil brings him a green powder. It explodes, and Plattner disappears.

Nine days later he returns by falling onto his headmaster in a strawberry bed. His heart now beats on the right, his liver and lungs are reversed, and he writes backwards with his left hand. Wells has put the whole man through the glove's turn. This is fiction; the reversal is no report of a physical experiment.

The narrator explains the geometry: moving a solid around in ordinary space cannot exchange its right and left sides. To do that, he proposes taking the body out of that space and turning it somewhere beyond. Wells gives Plattner a chemical mishap to get him there.

The narrator also has an awkward piece of evidence to explain. Photographs of Plattner at five and twenty-one show the reversed asymmetry of his face. A picture taken at fourteen seems to disagree. It was a cheap "Gem" photograph made directly on metal, which reverses things as a mirror does.

Earlier, in 1878, the physicist James Clerk Maxwell had used the extra dimension for a joke in "A Paradoxical Ode." His soul is an entangled knot, and the equipment for undoing it is elsewhere. "Since all the tools for its untying / In four-dimensioned space are lying," he writes.

Ana and kata

Charles Howard Hinton, a schoolmaster and writer of the 1880s, gave the missing directions names. He borrowed Greek words meaning roughly up and down: ana and kata. They name the two ways along the fourth spatial axis, just as we need two directions along any line.

In *A New Era of Thought*, published in 1888, Hinton defines ana as +W and kata as −W. Using the lowercase w of this book, going ana means moving in the positive w direction; going kata means moving in the negative w direction. We can name where the glove would go even though we cannot point that way.

Those words give this book its title. They also leave us with a question: how much of the motion we can describe can we learn to picture?

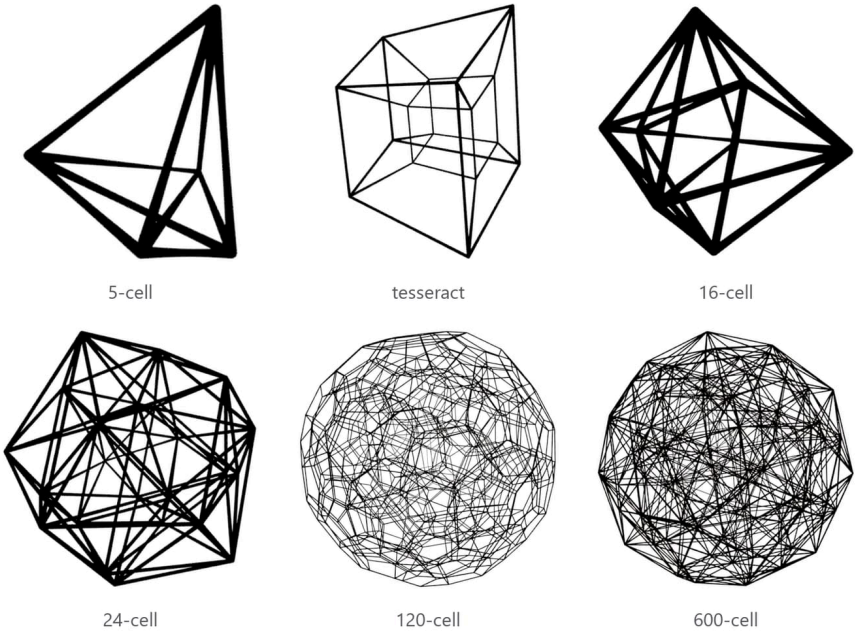
Hinton coined another word we will need: tesseract, the four-dimensional cube. He first spelled it "tessaract" in the 1888 book; by his 1904 book *The Fourth Dimension*, he used "tesseract." The name comes from Greek words for four and ray: four edges leave each

corner. The opening render shows that object. We will learn to picture it through images that bring it into fewer dimensions. Chapter 11 takes up Hinton's life.

What this book is

Part I starts with learning to picture four-dimensional shapes through their shadows and slices. A projection is a shadow; a cross-section is a slice. Both give us pictures to work with, though neither is the whole object. Learning to read them is part of the work, much as the paper hand helped us follow a turn we could not give a glove.

Part II follows the people who invented higher space. We will meet the six regular polychora, the regular four-dimensional solids. The comparison below shows each in stereographic projection, a way of making a three-dimensional shadow of a four-dimensional shape.

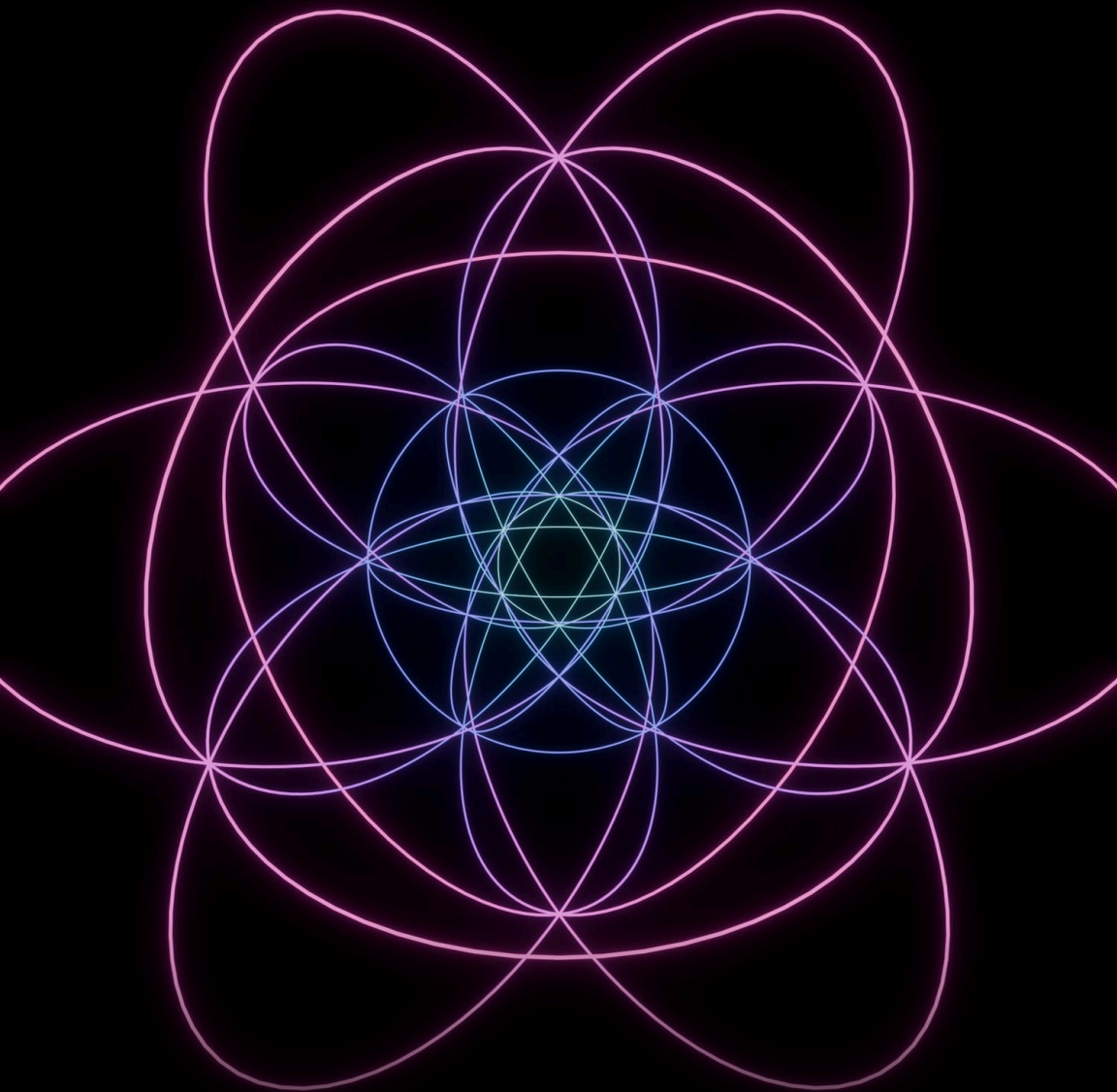


The six regular four-dimensional solids in perspective projection: 5-cell, tesseract (8-cell), 16-cell, 24-cell, 120-cell and 600-cell. These three-dimensional shadows introduce the shapes you will learn to picture.

Part III takes up the strange spaces mathematicians found. Part IV turns to time and the hidden dimensions of physics. Here we need to keep two meanings of the fourth dimension separate. Möbius's turn and Hinton's directions belong to a spatial fourth dimension, an extra direction at right angles to our three. Spacetime treats time as a fourth coordinate alongside space. Chapter 19 takes up that second meaning.

Part V follows the idea into ghosts, art and stories. Part VI turns to the mind and what we can learn to picture. Some of what follows is settled mathematics. Some familiar claims are contested or wrong; others are speculation, and the prose labels each accordingly. The next section, "How to Read This Book," explains the three evidence tiers and the figures.

You can put the gloves back on the table now. You still cannot give a rigid left glove the turn Möbius described, and nobody has a physical trick for doing it. We can follow the geometry by starting somewhere easier to stand outside. In Chapter 1, a square meets a sphere.



The 24-cell in stereographic projection, centred on one of its octahedral cells and seen down a three-fold axis. Its straight edges appear as arcs.

How to Read This Book

A small cube sits inside a larger one, their corners joined by lines: the familiar drawing labelled "tesseract." You are looking at a perspective shadow of a four-dimensional object. This guide explains the terms and picture labels you will meet in the book's figures.

Three kinds of claim

The book sorts factual claims into three evidence tiers. The wording tells you which one you are reading.

Established claims are checkable mathematics, physics or history, stated plainly without hedging.

A contested claim is widely repeated but overstated, disputed or wrong. The book names the problem and corrects it inside the story, explaining what the evidence supports.

Speculative ideas belong to philosophy or fringe science that geometry inspires but cannot establish. Phrases such as "No one knows whether..." flag these passages. They also say what would count as evidence for the idea.

Within an otherwise established chapter, a speculative passage may be set apart and labelled "Speculative." Some chapters lean toward one tier throughout: Chapters 1 to 3 are almost entirely established; Chapter 29, "The Speculative Frontier," is speculative throughout and marked accordingly.

The companion book *The Essence of Sacred Geometry* uses the same three-tier standard.

The book makes no health, therapy or wellness claims about studying geometry, and treats sacred diagrams such as the Tree of Life and the Sri Yantra as history and metaphor, without claiming they are hidden four-dimensional objects.

Names for the new direction

Coordinates are numbers that locate a point. In four-dimensional space we write them as (x, y, z, w). The fourth axis, w, is at right angles to the other three. Here w is a direction of space. Charles Howard Hinton called its two directions ana, toward +w, and kata, toward −w, using words he took from Greek.

A four-dimensional polytope is called a polychoron; the plural is polychora. Its boundary is made of three-dimensional cells, just as a cube's boundary is made of squares. Each cell is a polyhedron, a solid bounded by flat polygonal faces. Faces are two-dimensional polygons; edges are the lines where faces meet; vertices are corners.

A tesseract has eight cubic cells, 24 square faces, 32 edges and 16 vertices. All eight cubes are identical and meet at right angles. In the familiar drawing, perspective makes the far cell look like a little cube inside the larger one. Six other cubes become the lopsided shapes around it.

There are exactly six regular polychora. Their names throughout this book are 5-cell, tesseract (8-cell), 16-cell, 24-cell, 120-cell and 600-cell. The number tells you how many cells the shape has. The Swiss mathematician Ludwig Schläfli first found the six around 1852; Chapter 9 tells their story.

In four dimensions, rotation takes place in a plane; the familiar idea of turning around an axis no longer describes it. We name a coordinate plane by its two axes. There are six: xy, xz, xw, yz, yw and zw.

In three dimensions there are three coordinate planes and three axes, so describing rotation around an axis works there. In four dimensions the six planes outnumber the four axes. Read the pair of axis letters in a caption to find the plane of rotation.

For example, "rotated xw 45°" means a turn of 45° in the plane of the x- and w-axes. The turn changes x and w; every point with x = 0 and w = 0 stays put. That leaves a whole plane of points fixed. A double rotation turns in two completely perpendicular planes at once, at independent rates, leaving only the centre point fixed.

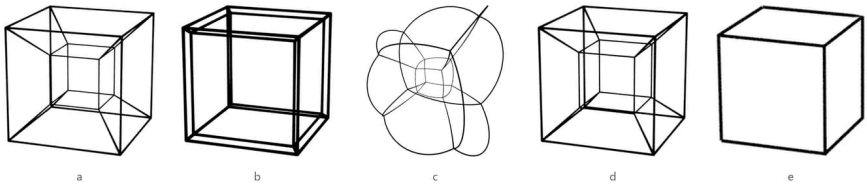
Reading the pictures

First check whether a picture shows a projection or a cross-section.

A projection is a shadow: the whole object flattened down one dimension. Parts that were apart can overlap, and lengths and angles can change. A cross-section is a slice through one chosen three-dimensional space. It preserves the geometry of the points in that slice and leaves out everything outside the cut.

The legend below shows one tesseract five ways. Each label tells you what the picture shows and what it distorts or leaves out.

- Perspective: a view from a point along w, like a photograph. Nearer parts look bigger and farther parts smaller; straight edges stay straight.
- Orthographic: a parallel shadow along w, without perspective shrinking. Near and far cells can coincide, making a cell-first tesseract look like one plain cube.
- Stereographic: the shape is puffed onto a hypersphere, a round surface in four-dimensional space, then projected from a point on it. Edges become circular arcs and cells near that point balloon outward; angles are preserved.
- Cell-first Schlegel: a perspective view from just outside one cell. That cell becomes the outer boundary, enclosing the others without overlap. You can follow their connections, though their shapes are distorted.
- Cross-section at w = c: a slice at one chosen value, c, of the fourth coordinate. It preserves the slice's geometry and omits the rest; w = 0 gives the cube shown here.



One tesseract, five pictures. (a) Cell-first perspective, (b) cell-first orthographic, (c) stereographic, (d) cell-first Schlegel diagram, (e) cross-section at w = 0.

Curved edges in a stereographic picture come from the projection; the polytope's edges are straight. A strip of cross-sections at different values of w shows the slices a three-dimensional observer would encounter as the object passed through their space along w. In Chapter 1, the inhabitants of *Flatland* watch the lower-dimensional version: a sphere passing through their plane.

The page is flat, so every four-dimensional figure is a two-dimensional image of a three-dimensional projection or slice. With a projection, you are looking at a shadow of a shadow. Chapter 3 explains how each view works and gives its history.

Plates, diagrams and exercises

Dark, full-bleed plates open the chapters, while black-on-white diagrams sit within the text. I rendered every polytope figure in Blender using my 4D Polytope Engine add-on. Other subjects, such as gloves and spheres passing through planes, were modelled as ordinary Blender scenes in the same two styles.

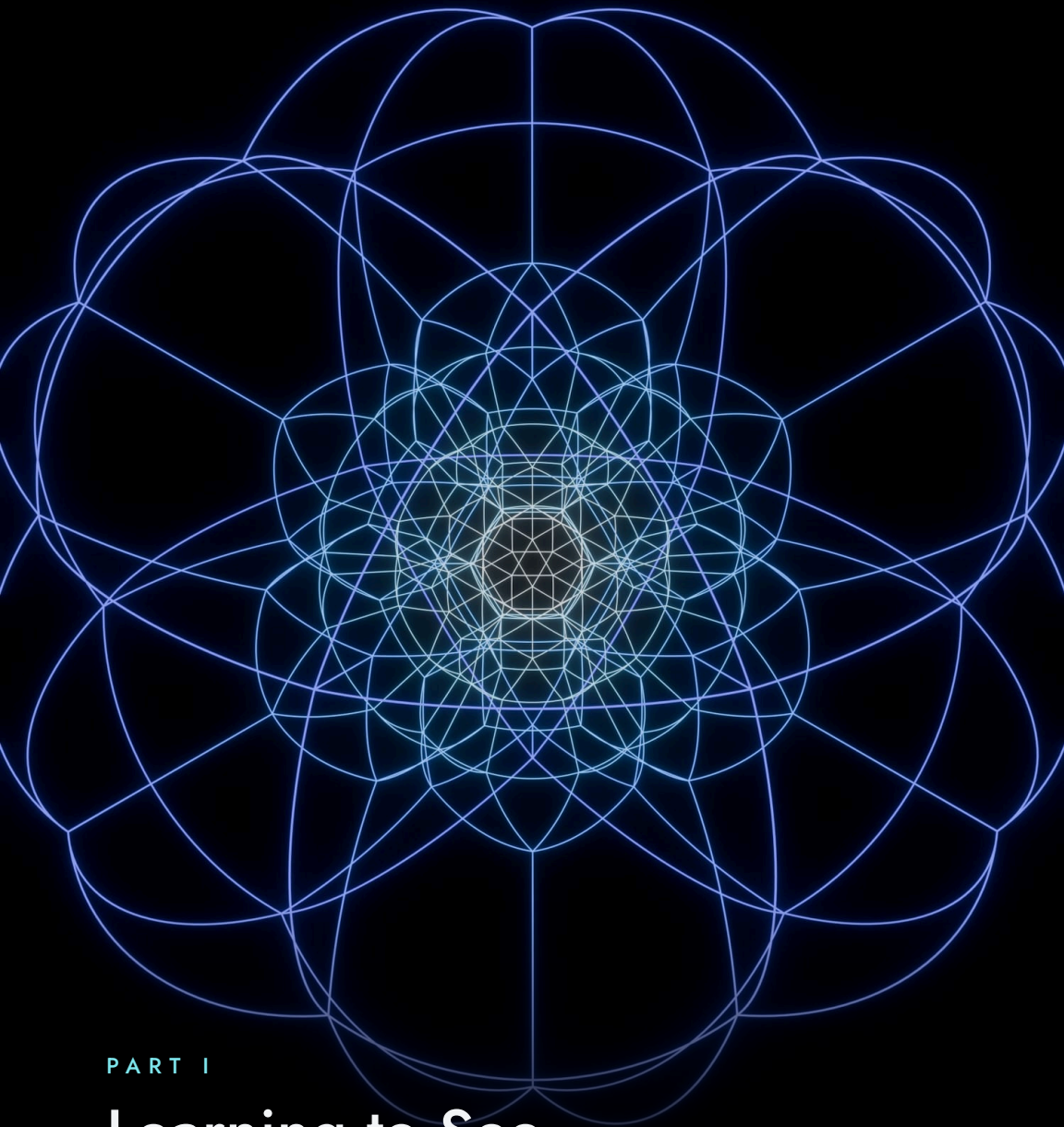
TRY IT In the viewer, choose the tesseract and press XY under Auto planes. The shadow simply spins, the way any solid turns. Press Reset View, then try XW alone: now the inner and outer cubes trade places. A turn in a plane that contains w is the motion we have no everyday word for.

The back matter has graded practice sessions and a data appendix for the mathematical details. You can check terms in the glossary and follow every factual claim to a source in the bibliography.

Dipping in

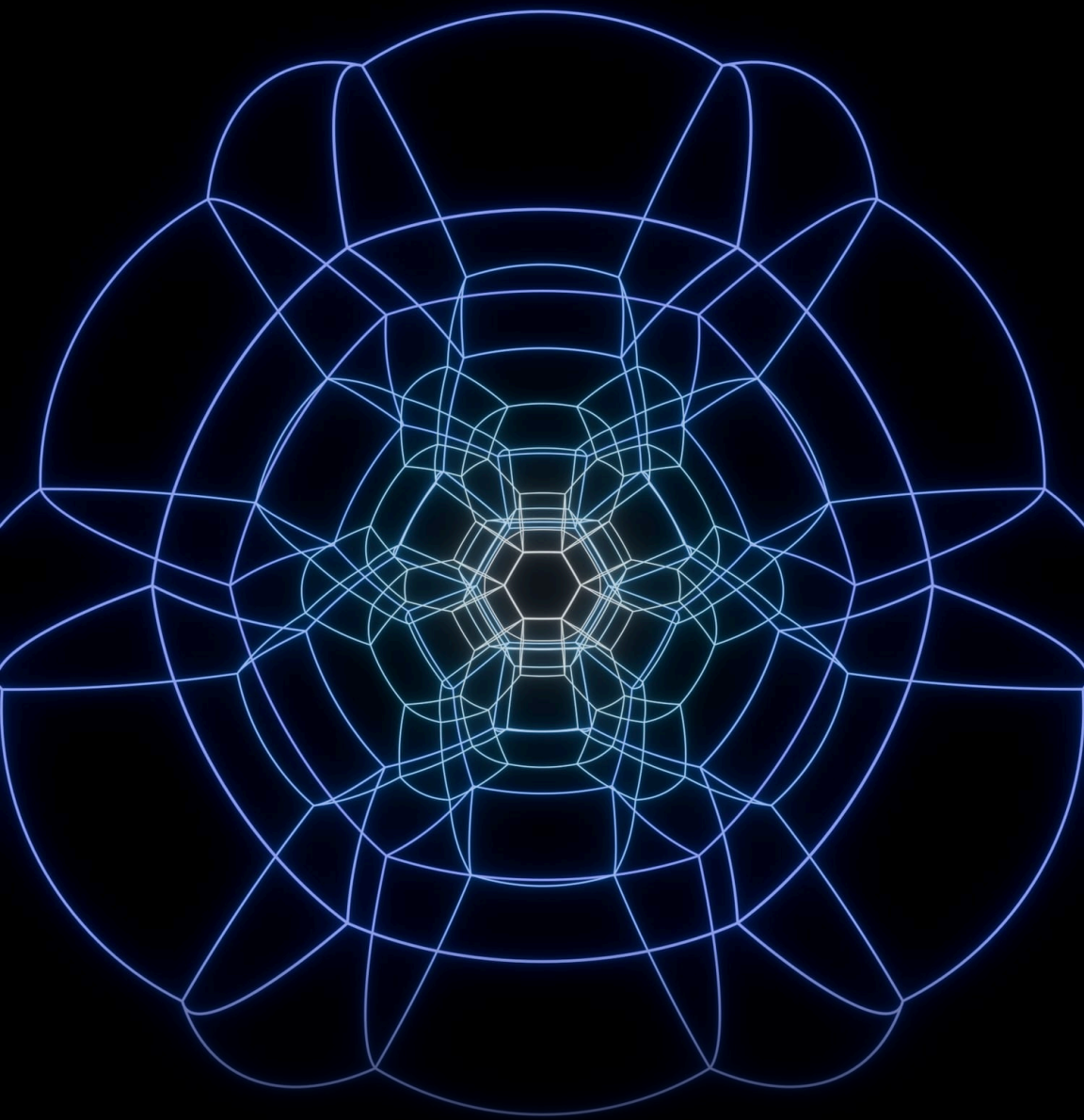
The 30 chapters are arranged in six parts, and each stands on its own. Start with a question that catches your interest. When a topic needs more room, a short cross-reference points to its full treatment elsewhere.

To begin reading in chapter order, turn to Chapter 1 and *Flatland*. How does a visitor with an extra direction make itself known to people confined to a plane?



PART I

Learning to See



The truncated 24-cell in stereographic projection, centred on one cell. Its 48 cells are 24 truncated octahedra and 24 cubes.

CHAPTER 1

A Square Meets a Sphere

The eve of the millennium

A Square sits with his wife on the last day of the 1999th year of Flatland's era, waiting to see the old millennium out. The lawyer has had a trying evening with his grandson.

The boy is a Hexagon, "a most promising young Hexagon of unusual brilliancy and perfect angularity." His grandfather has been explaining squares. Nine little squares, each an inch on a side, fill a square three inches across. That is what 3^2 means in geometry: three times three square inches.

"I suppose 3^3 must mean something in Geometry; what does it mean?"

"Nothing at all," his grandfather replies, "not at least in Geometry; for Geometry has only Two Dimensions."

The boy keeps going. A point moving three inches makes a line. A line moving parallel to itself makes a square. Surely a square "moving somehow parallel to itself (but I don't see how) must make Something else (but I don't see what) of three inches every way."

"Go to bed," says the Square, adding that "if you would talk less nonsense, you would remember more sense."

Near midnight, turning his half-hour glass for the last time in the old millennium, he returns to the matter. "The boy is a fool," he says aloud. At once he becomes "conscious of a Presence in the room."

The scene comes from *Flatland: A Romance of Many Dimensions*, published in London by Seeley & Co. in 1884. Its author and

illustrator was a London headmaster writing under the name A Square. The A is an article, as in a square drawn on paper.

The title page puts Shakespeare to work twice. From *Hamlet*: "O day and night, but this is wondrous strange"; from *Titus Andronicus*: "Fie, fie, how franticly I square my talk!" Even before the story begins, the author is enjoying himself.

The dedication addresses "The Inhabitants of SPACE IN GENERAL / And H. C. IN PARTICULAR" and hopes for "the possible Development / Of that most rare and excellent Gift of MODESTY / Among the Superior Races / Of SOLID HUMANITY". The reader belongs to solid humanity; H. C. gets a dedication to himself.

The author distributed a pre-publication issue in October. A thousand-copy first edition followed in November, then a revised edition in December with corrections to the text and illustrations and a new preface by the supposed Editor. That preface had to answer readers who found the inhabitants' treatment of women rather hard to stomach.

A country of lines and polygons

Flatland is a plane, a space with length and width and no thickness. Its inhabitants are flat figures moving within it. We can picture their outlines from above; they see one another edge-on, as lines. A Square cannot look down on a Hexagon and count his sides.

Recognizing a stranger takes practice. Voice helps a Flatlander place someone, and feeling an angle gives a clue to the stranger's shape. The upper classes also learn to judge how quickly edges fade into the fog, using differences in dimming to work out the shape they cannot see from above. Fog, the narrator explains, is "a blessing scarcely inferior to air itself."

Shape determines rank. Soldiers and workmen are isosceles triangles, with two equal sides; tradesmen are equilateral triangles, with all three sides equal. Lawyers are squares. Above them come pentagons and hexagons, with five and six sides, then polygons with so many sides that they pass for Circles. A polygon is a flat figure

bounded by straight sides. The hexagons are the lowest rank of nobility; the Circles are priests. Women are straight lines.

The narrator explains the inheritance rule with the same assurance he brings to geometry: a male child generally has one more side than his father. A Square's son is a Pentagon; the Pentagon's son is a Hexagon. Family advancement can be counted around the edges.

Listen to the Square long enough and the tidiness becomes unpleasant. A woman seen end-on is practically invisible, and a collision with her pointed end can kill. Flatland's response is the Female Code. Every house must have an eastern door reserved for women, who must enter "in a becoming and respectful manner". In public a woman must continually sound her Peace-cry, under penalty of death. A woman with a condition causing involuntary movements, even violent sneezing, must be destroyed.

The narrator reports all this as sensible administration. He is equally composed about an Irregular child, a polygon with unequal sides. If the Medical Board cannot cure the child, he recommends that it "be painlessly and mercifully consumed."

Colour once offered a way around the hierarchy: lower figures could paint themselves to pass as their betters. After the revolt called the Chromatic Sedition was suppressed, colour was banned. Shape had to remain legible as rank.

In a footnote, Abbott compares the women's door to a separate church entrance "for Villagers, Farmers and Teachers of Board Schools", reported in the *Spectator* in September 1884. The little geometric country has borrowed a doorway from England.

Whether the book's treatment of women succeeds as satire is a disputed judgment. Some early readers took the Square for a woman-hater. In the revised edition's preface, his supposed Editor replies that prison has changed the Square's views of women and the lower classes. Writing as a historian, he had identified himself " (perhaps too closely)" with Flatland's accepted opinions.

The Editor extends the comparison to historians in our world, "in whose pages (until very recent times) the destinies of Women and of the masses of mankind have seldom been deemed worthy of mention and never of careful consideration."

Abbott makes his defence through another fictional voice, which may leave a reader unconvinced. His own record gives us something else to weigh: he advocated improved education for women. Ian Stewart points to Abbott's daughter's education as evidence of his efforts to give women the same education as men. That sits uneasily beside the Square's calm account of the Female Code.

The headmaster behind the Square

The author's full name was Edwin Abbott Abbott. His parents, Edwin Abbott and Jane Abbott, were first cousins with the same surname, so Abbott became both their son's middle name and his surname. In 1865, aged twenty-six, he became headmaster of the City of London School. He stayed until 1889.

He was a clergyman and a scholar of classical literature and Shakespeare. His more than forty-five books included *A Shakespearian Grammar* and *How to Write Clearly*, as well as novels and theology. He was no mathematician: apart from a short passage in *Hints on Home Teaching*, *Flatland* was his only mathematical writing. At school, he made elementary chemistry compulsory in the upper classes.

The dedication's H. C. was Howard Candler, his lifelong friend and a mathematics master at Uppingham School. They had met as schoolboys at the City of London School and gone to Cambridge together. According to the mathematician Thomas Banchoff, they exchanged letters weekly during Candler's twenty-five years at Uppingham, though the letters have since disappeared. Abbott later identified Candler as the dedicatee in a theology book written after his friend's death in 1916, Banchoff records.

Charles Howard Hinton also taught at Uppingham, from 1880 to 1886, and had published an essay called "What is the Fourth

Dimension?" in 1880. Candler was a likely route by which Hinton's ideas reached Abbott, though there is no direct evidence that Abbott and Hinton met. Chapter 11 follows Hinton's own work.

In 1871 Abbott was invited to tea at the home of George Eliot and G. H. Lewes on the day the physicist John Tyndall came to dinner. Tyndall knew Hermann von Helmholtz, who lectured about imaginary flat creatures living on a surface. Stewart suggests that Tyndall may have told Abbott about them.

Upward, not northward

The night before the visitor arrives, the Square dreams of Lineland. Its men are lines and its women points, all confined to a single line. Its Monarch can conceive of movement only along that line. The Square tries to explain sideways and fails. He can move in a direction the Monarch cannot even point toward, yet he cannot make the Monarch understand it.

Back in his own house, the Square meets the same difficulty from the other side. His guest is a Sphere, a solid ball from three-dimensional Spaceland. Only the portion of the Sphere crossing Flatland's plane is present in the Square's world. That portion is a circle, which he sees edge-on.

As the Sphere passes through, the encounter starts at a point. The circle grows until the Sphere's middle reaches the plane, then shrinks to a point and disappears. This is a cross-section, a slice where the solid meets the plane. The sequence below pairs that slice with the flat observer's view of it.

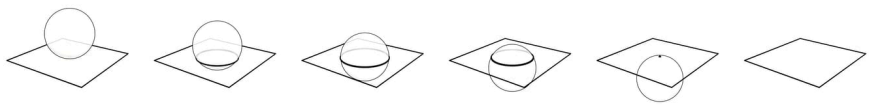


Figure 1.1 The Sphere passes through the plane: a point, a growing circle, the widest circle, a shrinking circle, a point, then nothing.

The Sphere can see into places the Square regards as closed. He lists "your houses, your churches, your very chests and safes, yes even your insides and stomachs, all lying open and exposed to my view." Walls that enclose a flat room leave its contents exposed from above.

Told he would need an eye on his side to see into Space, the Square protests: "An eye in my inside! An eye in my stomach! Your Lordship jests."

The Sphere descends into a locked cupboard and removes an account tablet without moving the door. He offers "just the least touch, in your stomach," and the Square feels "a shooting pain in my inside." Eventually the Sphere lifts him out of the plane altogether. The Square can now look into the world whose walls had confined his sight.

The direction is "Upward, not Northward". The Square repeats the phrase afterwards, trying to hold on to a distinction his ordinary experience keeps erasing. North is a direction within Flatland. Up carries him out of it.

Abbott's worlds form a ladder: Pointland has no dimensions, Lineland one, Flatland two and Spaceland three. The Square has failed to explain the second dimension to Lineland's Monarch and struggled to grasp the third himself. Once he accepts it, he wants to ask about a fourth.

Sixteen corners

The Square has seen a cube, the solid made when a square moves in the new direction. He returns to his grandson's unfinished thought, now in much grander language, and asks what moving a cube in another new direction would produce.

"In Three Dimensions, did not a moving Square produce—did not this eye of mine behold it—that blessed Being, a Cube, with EIGHT terminal points?"

Terminal points are corners. A line has two ends, a square four corners and a cube eight. The Square continues: "shall not, I say, the motion of a divine Cube result in a still more divine Organization

with SIXTEEN terminal points? Behold the infallible confirmation of the Series, 2, 4, 8, 16: is not this a Geometrical Progression?" Each number doubles the one before it.

He also counts the parts that bound a shape. A line is bounded by two points, a square by four lines and a cube by six squares. The proposed next shape "must have 8 bounding Cubes". Its boundary would be made of solids, just as a cube's boundary is made of squares.

The Sphere replies: "It is reported so. But men are divided in opinion as to the facts. ... no one has adopted or suggested the theory of a Fourth Dimension. Therefore, pray have done with this trifling, and let us return to business."

The Square keeps going, asking about a fifth dimension, then a sixth, seventh and eighth. His teacher silences him "in his voice of thunder" and drops him back into Flatland. The grandfather who sent a boy to bed for asking about cubes has found a teacher with a similar limit.

Later, in a dream, the Sphere shows him Pointland. Its only inhabitant is a Point that believes itself to be the whole universe. "Infinite beatitude of existence! It is; and there is none else beside It." The Sphere takes this as a lesson against being satisfied with what one knows, apparently untroubled by his own recent refusal.

The Square's numbers are right. A tesseract, the four-dimensional counterpart of a cube, has sixteen vertices, or corners, and is bounded by eight cubes; Chapter 2 does the counting. He has worked out those features of an unseen shape by extending the rule his teacher just taught him.

The Square's method is dimensional analogy: examine how a world with fewer dimensions relates to the next one, then apply that relationship one step higher. We can use his difficulty with a third direction to think about a fourth spatial direction of our own. By this geometric analogy, a four-dimensional visitor could see inside our rooms and bodies, remove something from a locked safe without

opening it, or vanish along the fourth direction; Chapter 5 develops those possibilities.

Carry the Sphere's visit one dimension up and a four-dimensional solid, such as the 600-cell, passes through our space instead. Its successive slices are three-dimensional shapes that grow and shrink. Chapter 3 examines slices and their difference from projections, the shadow views used to show higher-dimensional objects.

TRY IT Choose the 600-cell in the viewer and switch on XW under Auto planes. Its tetrahedra seem to bloom out of the centre and sink back in, though in four dimensions every cell keeps its exact size and shape. Only the shadow changes.

The Square pays for trying to spread what he has learned. Following a proclamation against the "Gospel of Three Dimensions", he is imprisoned. "Seven years have elapsed and I am still a prisoner." His brother saw the Sphere in the Council Chamber and still disbelieves in its existence. The narrator calls himself a "poor Flatland Prometheus", lying "in prison for bringing down nothing to my countrymen."

How it was received

In November 1884 the *Athenaeum* said the book seemed to have a purpose, though it was hard to discover what. In February 1885 the *New York Times* called it "a delirious book", likely to be "enjoyed by about six, or at the outside seven, persons in the whole of the United States and Canada". It particularly liked the unbelieving brother.

In the 1887 printing of his anonymously published theology book *The Kernel and the Husk*, Abbott recommended his own anonymous novel, along with Hinton's work. He gave Hinton the warmer recommendation: readers would do "still better" to study his "very able and original work". Abbott went on to explain how a four-

dimensional being, if one existed, could enter closed rooms and even bodies.

Hinton praised *Flatland* as ingenious, while explaining that its satire served a different purpose from his own investigation. "But we wish, in the first place, to know the physical facts." There is no evidence of hostility between them; each praised the other's writing in print.

The novel did not make Abbott famous in his lifetime. His *Times* obituary in 1926 omitted *Flatland*. The book has nevertheless never gone out of print. Its analogy also reached viewers of Carl Sagan's *Cosmos*.

In a 1920 letter to *Nature*, William Garnett read the Sphere's passage through the plane as a parable of time: the changes we experience might correspond to our space moving relative to a fourth dimension. Garnett had been Abbott's pupil. This was a later reading in terms of spacetime, which concerns space and time together. *Flatland* itself concerns spatial dimensions, so calling it a prediction of relativity gives the book credit for an interpretation Garnett supplied decades later; Chapter 19 takes up the distinction.

Flat worlds after Flatland

Abbott had predecessors, too. In 1846 the Leipzig physicist and psychologist Gustav Theodor Fechner wrote about a shadow living on a screen, unable to conceive of a direction out of it. Writing as Dr. Mises, he suggested that such a being would experience the third dimension as time.

Fechner had already used the same pen name in 1825 for *Comparative Anatomy of the Angels*, arguing that angels must be spheres, the most perfect shape. One chapter considered whether angels have legs; another called them living planets. Abbott's visitor is also a sphere, a pleasing coincidence with no demonstrated influence behind it.

Hinton's *A Plane World* put its inhabitants on the rim of a flat disk. They could move forward and back along the rim, and up and down;

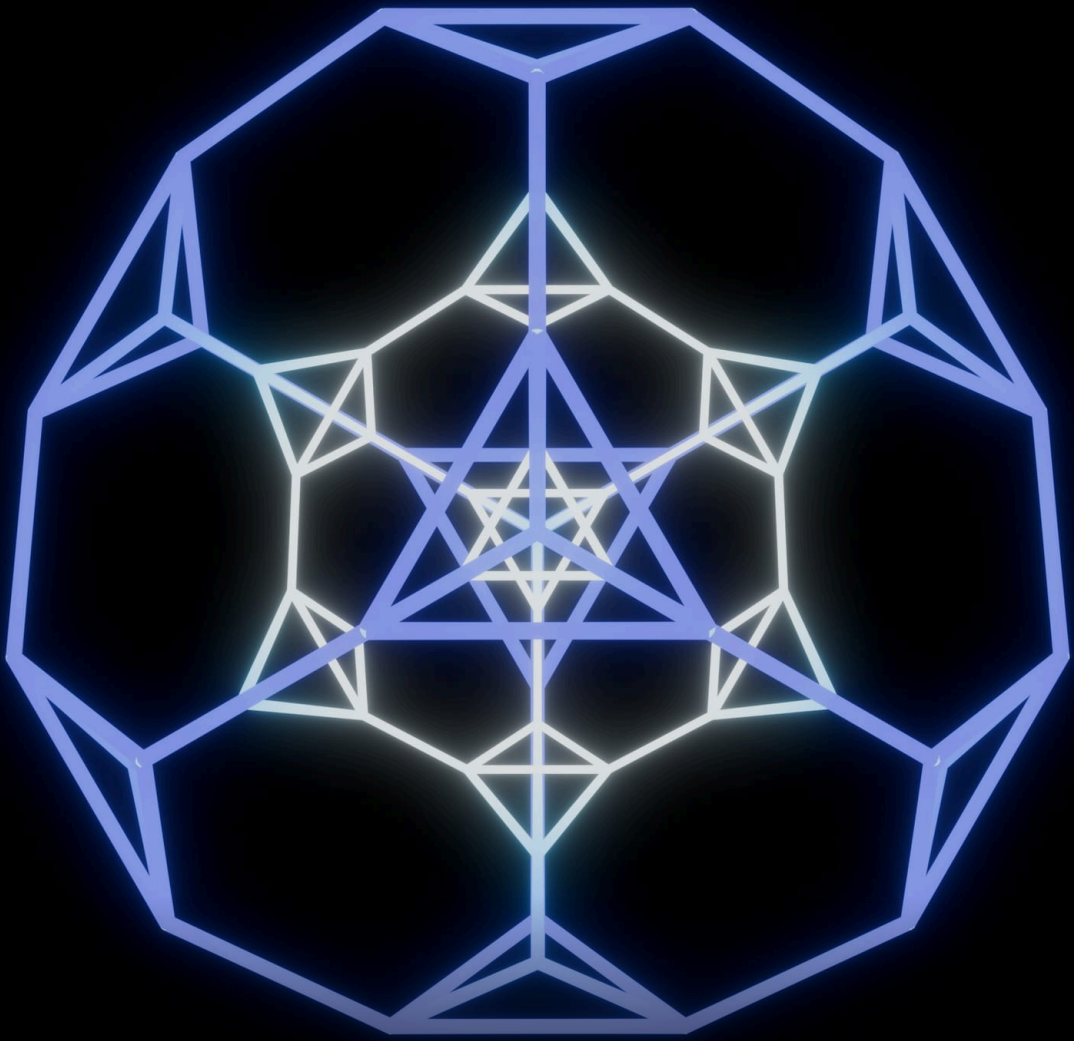
right and left were unknown to them. He wanted to work out the physical conditions of life in such a world.

Dionys Burger, a Dutch physics teacher, continued the Hexagon's story in *Sphereland* in 1957. A surveyor measures a triangle whose angles add to more than 180 degrees and is dismissed as a crank. He and the Hexagon work out that their world is the surface of a huge sphere, and later that their universe is expanding. Curved space returns in Chapters 8 and 12.

A. K. Dewdney's *The Planiverse* appeared in 1984, after years of work on two-dimensional science and engineering. He had begun working out the physics and chemistry of such a universe in 1977. The book's fictional frame has his students running a simulation called 2DWORLD and making contact with Yendred, an inhabitant of the disk-shaped planet Arde. Yendred is Dewdney spelled backwards. There, houses sit underground so rivers can pass over them. Nails are useless; things are joined with tape and glue.

Even digestion poses a problem. A channel running straight through a flat animal from mouth to anus would divide its body into separate pieces, so Arde's creatures digest another way; Chapter 5 returns to this problem one dimension up.

Chapter 2 returns to the Square's proposed solid. Starting with a point and following his sequence upward, we can count the sixteen corners and eight bounding cubes he described before his teacher sent him home.



The truncated tesseract in perspective: a tesseract with every corner cut away, leaving truncated cubes and tetrahedra.

CHAPTER 2

Counting Up

A cross of eight cubes

At the Metropolitan Museum of Art in New York, Christ floats in front of a cross made of cubes. There are no nails and no crown of thorns. Beneath him lies a chessboard floor; behind him is the bay of Port Lligat, where Salvador Dalí lived in Catalonia. At the lower left, Dalí's wife Gala looks up.

The canvas measures 194.3 × 123.8 centimetres. Dalí painted it in 1954, and it came to the Met as a gift from the Chester Dale Collection in 1955. Its title is *Crucifixion (Corpus Hypercubus)*. The last word offers a clue to that peculiar cross.

Four cubes stand in a column. At the second cube from the top, four more jut out, one from each of its exposed side faces. You can recognise a cross in those eight cubes, though the extra arms take some explaining. Why eight? What does this shape have to do with a fourth dimension?

The cross is a tesseract, unfolded. To understand what has been unfolded, leave the gallery for a moment and start with a point.

Drag and count

A point has no length. Move it in a straight line, keeping everything it passes through, and you get a line segment. Now move that segment sideways, at right angles to itself. If the distance you move it equals its length, it sweeps out a square.

Move the square at right angles to its plane, again through the same distance. The swept-out shape is a cube. This process is called extrusion: carrying a shape along a new direction and including all the positions it occupies along the way.

The shape doing the moving keeps its size and stays parallel to its starting position. The segment does this as it sweeps out a square, and the square does it as it sweeps out a cube. We can recognise the original pieces at the end of each move and follow what each has traced between its two positions.

Move the cube the same distance in a fourth spatial direction, at right angles to all three of its edge directions. The result is a tesseract (8-cell), the four-dimensional cube. The instruction still works even though we cannot picture the direction it asks for.

In *The Fourth Dimension*, published in 1904, Charles Howard Hinton put the difficult step this way: "If now the cube A moves in the fourth dimension right out of space, it traces out a higher cube—a tesseract, as it may be called."

In *Flatland*, the Square's grandson guessed the square-to-solid step we met in Chapter 1; the Square later extended the argument to a moving cube and a fourth dimension.

The sequence below puts the successive shapes side by side. The final picture is a perspective projection, a shadow of the tesseract; Chapter 3 explains how such pictures work.

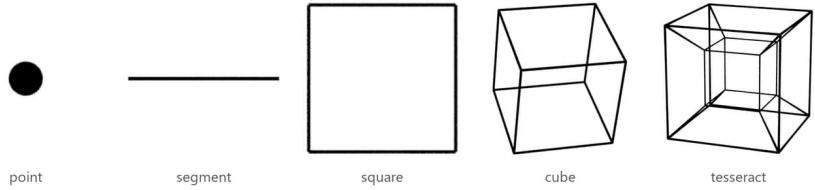


Figure 2.1 Drag a point to get a segment, a segment to get a square, a square to get a cube, a cube to get a tesseract.

To count the pieces, keep track of the starting and finishing positions. A segment has the original point at one end and a copy at the other. Between them, the moving point has swept out an edge.

When the segment becomes a square, its two endpoints each sweep out another edge. Add those to the original segment and its final copy: four edges. The endpoints themselves have doubled, giving four corners. Each drag leaves two copies of every old piece, and each old piece sweeps out a new piece one dimension higher.

When a cube makes the next move, its eight corners have eight copies at the end of the drag. The tesseract has 8 + 8 = 16 corners.

The cube also has twelve edges. Keep the twelve at the start and the twelve at the end. Each of the eight moving corners draws one more edge between them. That makes 12 + 12 + 8 = 32 edges.

There are six squares on the starting cube and six on the finishing cube. Each of the twelve old edges sweeps out a square, adding twelve more: 6 + 6 + 12 = 24 squares.

For the cubes, we have one at the start and one at the end. Each of the six square faces sweeps out another cube during the move, giving 1 + 1 + 6 = 8 cubes.

Hinton wrote: "The whole solid cube A is to be looked on merely as a boundary of the tesseract." Just as a square is a face of a cube, a whole cube is a boundary piece of a tesseract. These three-dimensional boundary pieces are called cells, which is why the tesseract is also the 8-cell.

Shape	Corners	Edges	Squares	Cubes	Tesseracts
Point (0D)	1				
Segment (1D)	2	1			
Square (2D)	4	4	1		
Cube (3D)	8	12	6	1	
Tesseract (4D)	16	32	24	8	1

The final 1 in each row counts the whole shape. The filled square sweeps out the filled cube, and the filled cube sweeps out the filled tesseract. Keeping those entries lets us use the same counting rule at every step.

Te counts confirm the Square's guesses in *Flatland*: sixteen corners and eight bounding cubes. Dragging also gives us the edges

and squares, without requiring a picture of the fourth direction.

At each corner, or vertex, of a tesseract, four edges meet. Every one is at right angles to the other three. Hinton's word tesseract comes from Greek words for "four" and "ray": four rays leaving a corner. Around each edge, three cubes meet. Every square is shared by two cubes.

Counting the six faces of all eight cubes separately would count each shared square twice. Dividing 8 × 6 by 2 gives 24, agreeing with the dragging rule, which counted each square once, including those joining the starting cube to the finishing cube.

Each cube shares its six square faces with six other cubes. There is one cube left over, its opposite, which it never touches. The eight cubes come in four opposite pairs. Compare that with an ordinary cube: each square face touches four others and never touches the square opposite it.

TRY IT Choose the tesseract, press Reset View, then rotate slowly with the A and D keys (the ZW plane). Try to count all sixteen corners. Some seem to merge and split apart as you turn; that is the shadow overlapping, not the object changing.

Sixteen addresses

Give a square's corners addresses using two numbers, each either 0 or 1. A cube needs three such numbers. For a tesseract, use four, written (x, y, z, w), with w naming the fourth coordinate. Each coordinate records a position along one direction.

The addresses begin at (0, 0, 0, 0) and end at (1, 1, 1, 1), with every combination in between. Written without commas, they are four-digit strings such as 0000 or 1011. Because each place has two choices, there are $2^4 = 16$ strings.

Two corners share an edge exactly when their addresses differ in one digit. From 0000 you can go straight to 1000, 0100, 0010 or 0001.

Getting to 1100 requires changing two digits, so that pair is not joined by an edge.

Read as a network, the tesseract is a collection of corners connected by that rule. From any address you have four neighbours, found by changing each digit in turn.

The addresses also locate the opposite cubes. Hold x at 0 and let the other three coordinates vary: those corners belong to one cube. Hold x at 1 and you have its opposite. The same choice can be made for y, z or w, giving four pairs. The two cubes in a pair cannot touch because no point can have both values of the fixed coordinate.

Sort the addresses by the number of 1s they contain. Only 0000 has none; four addresses have a single 1. Six have two, four have three, and only 1111 has four. The counts are 1, 4, 6, 4, 1, a row of Pascal's triangle. These five groups mark the successive layers from one corner to its opposite; Chapter 3 follows them in corner-first slices through the tesseract.

A walk along the edges that visits every corner once, changing one digit at a time and returning to its starting corner, gives a Gray code, an idea described for electronics in patent applications in the 1940s.

Engineers have used the same network rule in higher dimensions to wire computers. The Caltech Cosmic Cube had 64 processors connected as the corners of a six-dimensional cube, each communicating directly with its six neighbours. The wires could sit in an ordinary workshop; only their connections had to follow the six-dimensional rule.

The Connection Machine had up to 65,536 processors, grouped sixteen to a chip on 4,096 chips. Those chips were wired as a twelve-dimensional cube, each chip connected directly to twelve others.

Tamiko Thiel, who led the machine's design, gave it a case made of eight smaller cubes arranged into a cube, standing 1.5 metres tall. The case showed the hypercube idea on the outside, in three-dimensional space, while the chips inside followed the connections of a twelve-dimensional cube.

The formula

To count squares in a cube of any dimension, we can choose their directions and their positions. That gives us a formula we can use without working through every earlier drag.

For this purpose, "face" means a piece of any dimension, including a corner or an edge. Let n be the dimension of the whole cube, and k the dimension of the pieces you want to count. For squares in a tesseract, n is 4 and k is 2.

The number of k-dimensional faces is:

2^(n−k) × C(n, k)

C(n, k) means the number of ways to choose k directions from the n available. To specify a square in a tesseract, choose the two directions it runs along. There are six pairs: xy, xz, xw, yz, yw and zw. For each remaining direction, choose which end it sits at, 0 or 1. Those two fixed directions give four end combinations: 00, 01, 10 and 11. Six times four gives twenty-four squares. In the general formula, 2^(n−k) counts these end choices for the n − k fixed directions.

The same rule counts the cubes. Choose three of the four directions for a cube to run along, then choose an end in the remaining direction. With four choices of directions and two choices of end, we get eight cubes.

We can collect all the counts by multiplying out $(x + 2)^n$, which means multiplying n copies of (x + 2) together. Here x is a counting label, separate from the coordinate x in a corner's address. The number multiplying x^k tells you how many k-dimensional pieces there are. That multiplying number is called a coefficient.

For a tesseract, the expression is:

$(x + 2)^4 = x^4 + 8x^3 + 24x^2 + 32x + 16$

Read it from the right. The 16 counts corners. The 32 multiplying x counts edges; the 24 multiplying x^2 counts squares. Then come eight cubes and one whole tesseract. The power of x tells you the dimension of the piece: x^2 means x multiplied by itself, and x^3 means three factors of x.

Each of the four directions contributes one factor, $x + 2$. A face either runs along that direction, represented by x, or sits at one of its two ends, represented by 2. Choosing a term from each factor specifies all four choices. Two free directions give x^2, so the terms collected beside x^2 count the squares. The same reasoning works for n directions.

You can also multiply out $(2x + 1)^n$. For four dimensions this gives:

$(2x + 1)^4 = 16x^4 + 32x^3 + 24x^2 + 8x + 1$

The counts run in the opposite order. Now the power records how many directions are fixed. A corner has all four fixed, so the coefficient of x^4 is 16. A cube has one fixed direction, so the coefficient of x is 8. The final 1 counts the whole tesseract, with no direction fixed.

For comparison, $(x + 2)^3$ gives $x^3 + 6x^2 + 12x + 8$. Read it in the same way: one whole cube, six square faces, twelve edges and eight corners. The expression includes the whole cube because a face, in this broad counting sense, can have the full dimension of the object. That is the role of the leading x^4 in the tesseract's expression too.

For a five-dimensional cube the rule gives 32 corners, 80 edges, 80 squares, 40 cubes and 10 tesseracts. The next drag has made the whole tesseract into a boundary piece, just as the previous drag did to the cube.

We can check the counts by subtracting edges from corners and adding faces. For an ordinary cube:

$8 - 12 + 6 = 2$

This is Euler's formula. It holds for convex polyhedra, the solid shapes with flat faces and no inward dents. In four dimensions, the alternating count needs another term, the cubes:

$16 - 32 + 24 - 8 = 0$

The cubes are part of the boundary here, so they need their turn in the sum. It is also zero for each of the six regular polychora, the regular four-dimensional shapes; Chapter 9 introduces the full set.

We can measure the tesseract too. A unit shape has edge length 1. The longest diagonal of a unit square is $\sqrt{2}$; for a unit cube it is $\sqrt{3}$.

The tesseract's longest diagonal is $\sqrt{4}$, which is exactly 2. In general, a unit n-dimensional cube has a longest diagonal of $\sqrt{n}$.

Its centre lies halfway along that diagonal. The distance from the tesseract's centre to any corner therefore equals its edge length. Among cubes of positive dimension, the tesseract alone has this property. It also occurs in a few other shapes.

The amount of four-dimensional space inside is called its hypervolume. If an edge has length s, the hypervolume is s^4: multiply four copies of s. Doubling the edge multiplies the content by 16. For cubes in n dimensions, doubling the edge multiplies the content by 2^n.

We can also build families of shapes by adding points. Add one new point and join it to every existing corner, using a new dimension at each step. This is the simplex ladder: point, segment, triangle, tetrahedron, then 5-cell. Joining a point outside a triangle's plane to its corners makes a tetrahedron. The next step takes us into four dimensions: the 5-cell has 5 corners, 10 edges, 10 triangles and 5 tetrahedral cells.

The cross-polytope ladder starts with a pair of points on an axis, a line through the centre along one direction. Add a pair on each new axis, one on either side of the centre, and join every point to every other except its opposite partner. This gives a segment, a square seen as a diamond, an octahedron, then a 16-cell. The octahedron is the shape reached with pairs on three axes.

The 16-cell has 8 corners, 24 edges, 32 triangles and 16 tetrahedral cells. Those are the tesseract's numbers in reverse order. The pair are called duals: placing a point at the centre of each of the tesseract's eight cubes gives the eight corners of a 16-cell. The comparison below shows the three four-dimensional shapes side by side.

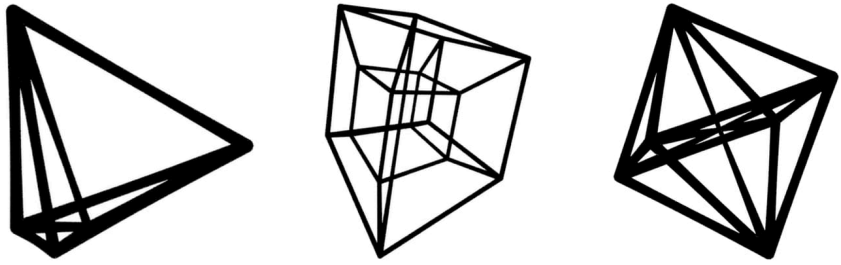


Figure 2.2 The 5-cell, tesseract and 16-cell in vertex-first perspective projection. The tesseract and 16-cell have the same counts of boundary pieces in reverse order.

From five dimensions upward, these three families supply all the convex regular shapes, whose matching parts fit together alike throughout; Chapter 9 tells the story of that restriction.

Unfolding

Back at Dalí's painting, we can account for the eight cubes. To account for their arrangement, begin with a cardboard box.

Cut some of the edges of a cube's boundary so that its six square faces can lie flat without overlapping. The resulting pattern is called a net. A cube has eleven different nets. One is a Latin cross: a column of four squares with two arms attached.

To fold that cross into a cube, the column wraps into a ring of four faces, and the two arms close its open ends. While lying flat, the squares occupy a plane. Folding takes them into the third dimension.

The Dalí cross repeats the construction one dimension higher. Its column of four cubes wraps into a ring through the fourth spatial direction, and its four arms fold in to close it. As the squares fold out of a plane to make the boundary of a cube, the cubes fold out of our space to make the boundary of a tesseract.

Every square face of every cube then meets another cube. Some of those joins survive in the net, where cubes remain attached face to

face. Others have been cut open to let the boundary unfold. When it folds, those separated faces must meet again, just as the cut edges of a cardboard box meet when you put it back together.

There are exactly 261 tesseract nets. Each consists of eight cubes arranged face to face in three-dimensional space, without overlaps. Dalí's cross is one choice among them.

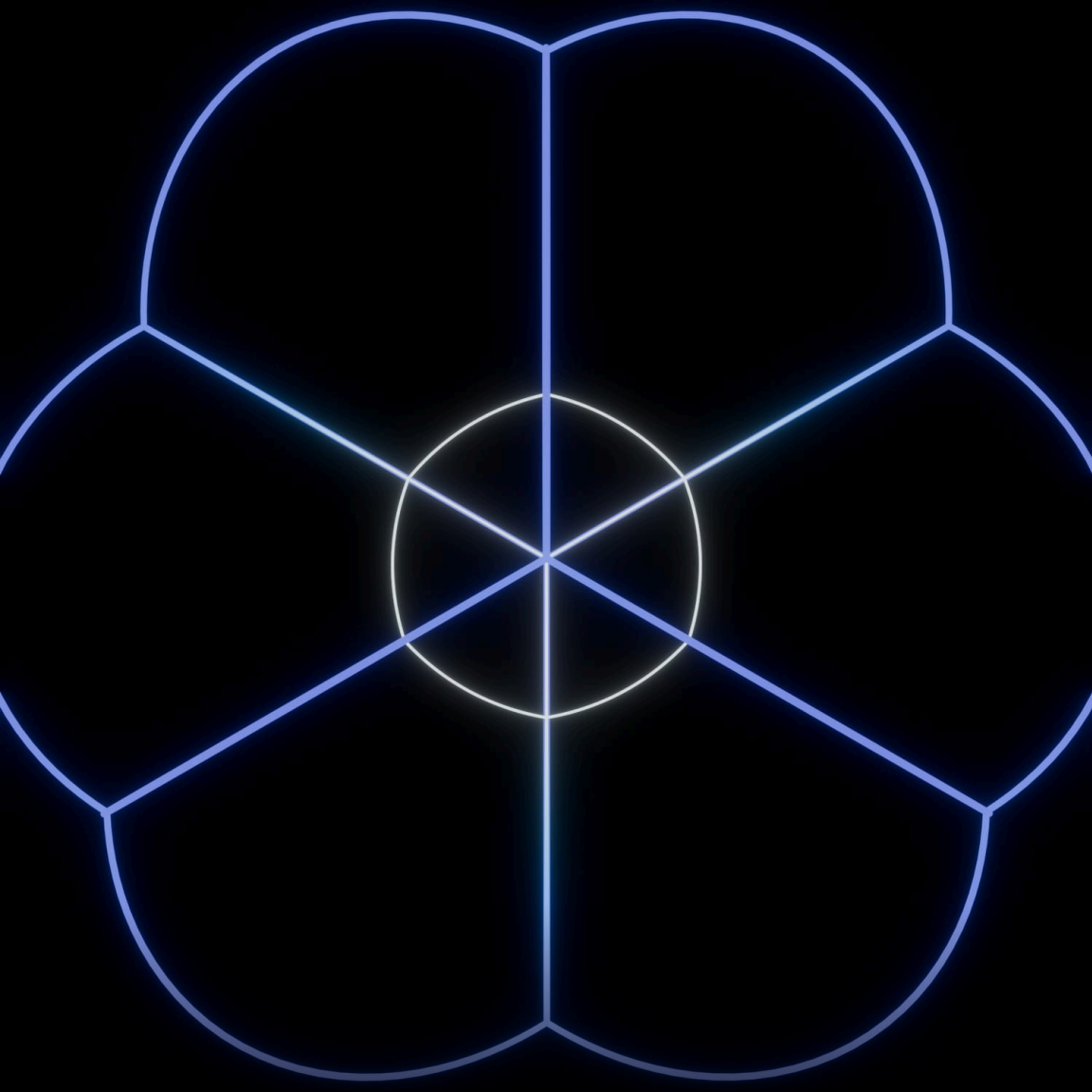
In his November 1966 *Scientific American* column, Martin Gardner asked how many different arrangements of eight cubes could result from unfolding a hollow four-dimensional cube. He did not know the answer. Peter D. Turney supplied 261 in *Unfolding the Tesseract*, published in the 1984–85 volume of the *Journal of Recreational Mathematics*.

Nets themselves had been in print much longer. The earliest known printed examples of nets of solids appear in Albrecht Dürer's 1525 book on measurement.

Copies of Dalí's eight-cube cross also fit together to fill three-dimensional space without gaps. Giovanna Diaz and Joseph O'Rourke showed this in 2015. The unfolded shape has something to do while it waits for a fourth direction.

In Robert A. Heinlein's 1941 story about an unfolded tesseract house, the architect Quintus Teal builds eight cubical rooms in this cross shape; Chapter 26 follows what happens to his house.

The cross in *Corpus Hypercubus* is a net, no more a tesseract than the flat cardboard cross is a box. Its cubes are laid out in our space; closing them requires the extra direction. Chapter 3 turns to projections, or shadows, and cross-sections, or slices: two other ways to study a shape that extends beyond the space we can see.



A tesseract in stereographic projection, centred on one cubic cell. The projection bends straight edges into arcs.

CHAPTER 3

Shadows and Slices

The hole in the panel

In Florence, around 1420, a viewer stands in the cathedral doorway with the back of a painted panel against his face. Filippo Brunelleschi has painted the Baptistery opposite and drilled a hole through the panel. The viewer looks through it and holds a mirror at arm's length. In the mirror sits the painted building. Lower the mirror, and the real Baptistery takes its place, at the same apparent size.

That is the experiment as Antonio Manetti and Giorgio Vasari tell it. The panel is lost, and historians dispute details of the account. Even the hole raises a problem: its roughly 15° field of view is narrower than the scene described. Brunelleschi is generally credited with these early perspective experiments, though we cannot reconstruct the whole scene from the surviving account.

The geometry is easier to pin down. Draw straight lines from the building to one eye. A flat surface cuts across those lines; where they meet it, you have the positions needed for the picture. The building has depth, while its picture has only width and height. Perspective gives us rules for making that picture.

Brunelleschi published no mathematics. Around 1435 his friend Leon Battista Alberti set out the method in *De pictura*. Piero della Francesca extended it in *De prospectiva pingendi* in the 1470s.

This is a projection: a way of representing an object in one fewer dimension. In perspective, the projecting lines run toward a single eye or viewpoint. In orthographic projection, they run parallel.

Both methods can bring a four-dimensional shape into three dimensions. Here the fourth dimension is spatial, an extra direction called w, distinct from time.

Our eyes already handle pictures that have lost a dimension. The retina is a two-dimensional array of receptors. The brain infers depth from clues, including shading and the differences between the two eyes' views. A photograph also puts a three-dimensional scene on a flat surface. We read it by making allowances for what the picture has done to its subject.

Recall the Sphere passing through Flatland in Chapter 1. The inhabitants first encounter a point, then a circle that grows to the Sphere's full width and shrinks back to a point. They see only the part lying in their plane. That is a cross-section, or slice.

A projection, which we can call a shadow, includes every part of an object, though parts may overlap and lengths and angles may be distorted. A cross-section keeps the true size and shape of the piece where the cut passes, leaving everything else out. The Flatlanders' circle was a slice, not a shadow. Those two words describe different operations throughout this book.

A hypersphere, the four-dimensional counterpart of a sphere, passing through our space would give us a similar experience: a point followed by a growing sphere, then a shrinking sphere and a point. At each moment, the rest of the object would lie outside the space available to us.

The cube that isn't inside a cube

Look straight at one face of a wireframe cube in perspective. The front face is a large square. The back face appears as a smaller square inside it, with corresponding corners joined. The lines between them are the cube's remaining edges.

Nobody takes this as evidence that a cube contains a miniature square. The back face is the same size as the front one; distance has made it look smaller. We also accept the sloping lines as a picture of

edges that meet at right angles. We know the object well enough to forgive its portrait.

Move the arrangement up one dimension. A tesseract is bounded by eight identical cubes, called cells. Look straight toward one cell in four-dimensional space, and project the result into three dimensions. This cell-first perspective view gives the familiar small cube inside a large cube, their corresponding corners joined.

The large cube is the cell nearest the viewpoint. The small cube is the opposite cell, farthest away. Between them sit six lopsided shapes, each running from a face of the outer cube to the corresponding face of the inner one. These are frustums, shapes like pyramids with their tops cut off. Each is the distorted image of another cubic cell.

Here the picture is harder to forgive. Both cubes occupy the three-dimensional space in which we are trying to understand the object. The smaller one looks thoroughly enclosed. Calling it an inner cube feels like a description of the tesseract itself.

All eight cells are equal cubes, and no cell sits inside another. Those six tapering frustums are cubes too. The edges meeting at a corner of the actual tesseract meet at right angles. The projection has changed apparent sizes and put separated parts inside one another's images.

Keep the wireframe cube beside this description in your mind. Its small square is the far face; the tesseract's small cube is the far cell. A face-on view hides or distorts five of the cube's faces around the nearest one. The cell-first tesseract view does the same to seven cells around the nearest cell. Changing the viewing direction or the rules of projection gives us other pictures to work with.

Four ways to cast a shadow

In orthographic projection, think of light arriving from infinitely far away, so its rays are parallel. Distance no longer makes things shrink. A cube seen face-on becomes a single square, because its front and back faces land on top of each other.

The cell-first orthographic view of a tesseract is equally plain: a cube. Its near and far cells coincide, while the other six flatten into the cube's six square faces. All eight cells contribute to the image, but several occupy the same place. You cannot tell them apart in this view.

Turn the objects corner-first and their shadows spread out. The cube's orthographic outline becomes a regular hexagon, a six-sided shape with equal sides and angles. The tesseract's becomes a rhombic dodecahedron: a solid bounded by twelve identical rhombuses, the slanted four-sided shapes with equal sides. It is a common crystal shape of garnet.

Inside this outline, the eight cubic cells appear as eight squashed cubes packed together. Two corners, the nearest and farthest, land on the same point at the centre, so you still cannot tell those corners apart. Of all the viewing directions, corner-first gives the tesseract's orthographic shadow its greatest volume.

The rhombic dodecahedron itself has fourteen corners and twenty-four edges. Copies of it can fill three-dimensional space without gaps. A shadow from four dimensions has given us a solid that packs neatly into our own space.

Other directions give different orthographic outlines. Seen face-first, the tesseract produces a box; seen edge-first, it produces a hexagonal prism, a solid with a hexagon at each end. Six cells become squashed prisms around its middle.

Architects' plans and elevations use orthographic projection. Gaspard Monge developed descriptive geometry, a system for describing solid objects through projections onto two planes at right angles. His *Géométrie descriptive*, compiled from lectures, appeared in 1799. Reading the two views together gives a draughtsperson information that either view alone would hide through overlap.

In perspective, a corner-first view of a tesseract gives a different surprise. Four distorted cubes surround a point at the centre of the projection. That central point is the corner nearest the viewpoint. Nearness in the fourth direction has put it inside the three-

dimensional image. Four cells are visible; the other four lie behind them in that extra direction, just as a corner-first view of a cube shows only three of its six faces.

The third method, a Schlegel diagram, is a particular use of perspective. Take a polytope, the general name for a shape of this kind, and place the viewpoint just outside one cell. Project through that cell. Its image becomes the outer boundary, containing the images of all the others without crossings.

For the tesseract, this produces the cube-in-a-cube arrangement again. Every cell's neighbours can be read from it, even though the shapes are distorted. Another example is the 5-cell: its Schlegel diagram is a tetrahedron divided into four smaller tetrahedra. A tetrahedron is a solid with four triangular faces. Count the outer cell as well as the four inside, and all five are accounted for.

Victor Schlegel introduced these diagrams in 1886. He taught at a Gymnasium, a school in Waren, a small town in Mecklenburg. Earlier he had been a colleague of Hermann Grassmann and became an early champion of Grassmann's mathematics.

The fourth method, stereographic projection, begins with a sphere. Choose a point on its surface and draw lines from it through the other points to a plane. Regions near the chosen point grow enormously in the image; the chosen point itself goes to infinity. The projection is conformal, meaning it preserves angles. Circles on the sphere become circles or straight lines on the plane.

For a four-dimensional polytope, first push its points outward onto a round 3-sphere centred on the object. A 3-sphere is the three-dimensional surface of a four-dimensional ball. Then project from a point on that surface into our space. Straight edges become circular arcs, and flat faces become pieces of spheres. Cells near the projection point swell; distant ones shrink toward the middle. The stereographic step keeps the angles on the spherical surface.

In the comparison below, the stereographic view's curved edges come from the way the image is made. The polytope's original edges

are straight. The curves in the picture give us no reason to think that the surrounding four-dimensional space is curved.

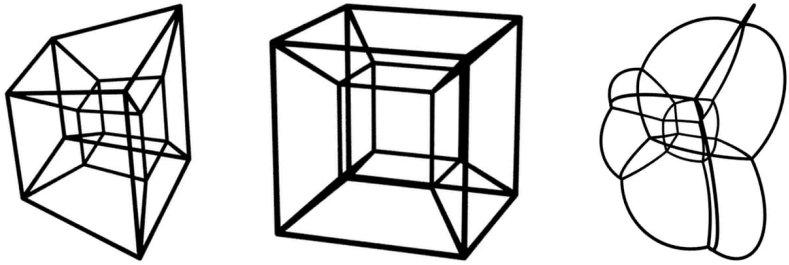


Figure 3.1 Perspective, Schlegel and stereographic views of the same tesseract. The Schlegel diagram maps connections; the stereographic view keeps angles on the spherical surface at the cost of straight lines.

The earliest surviving description of stereographic projection is in Ptolemy's *Planisphere*, from the second century AD, where it was used to map the sky. François d'Aguilon gave it its name in 1613, in a book illustrated by Peter Paul Rubens.

Thomas Harriot proved in the late sixteenth century that the projection preserves angles. He never published the proof, and it remained among his papers for more than three centuries. Edmond Halley published a proof in 1695.

For a large polytope such as the 120-cell, with its 120 dodecahedral cells, stereographic projection is often the easiest view to read. A dodecahedral cell has twelve faces. The cells can all appear at once, arranged in rings from swollen ones outside to small ones near the centre; Chapter 10 takes up the 120-cell itself.

Slicing

Push a cube face-first through a plane occupied by Flatlanders. A square appears, stays the same size throughout the passage, then vanishes. They have seen successive cuts through a cube, with no perspective shrinking involved.

Push it corner-first and a point arrives, followed by a small equilateral triangle, with three equal sides. The triangle grows until the cutting plane reaches three of the cube's corners. Continue the passage and the slice has six sides. Exactly halfway through the cube it is a regular hexagon. Later it becomes a shrinking triangle, ending at the opposite corner.

The same cube can supply a steady square or a changing succession of polygons, flat shapes with straight sides. The direction of passage determines which shapes the Flatlanders encounter. The cube itself stays the same shape throughout.

Now let our three-dimensional space be the cutting space. A tesseract moving cell-first through it produces a cube that appears, stays the same size and vanishes. The rest of the tesseract remains outside the slice. Our cube is as incomplete an encounter as the Flatlanders' square.

For the corner-first passage, begin with a single point. It grows into a regular tetrahedron. As the slice advances past the next group of corners, the tetrahedron's tips are cut away. The resulting shape is a truncated tetrahedron, with four triangular faces and four hexagonal faces. Its proportions change as the cut moves; at one stage all its edges are equal.

Exactly halfway through the tesseract, the slice is a regular octahedron, a solid with eight triangular faces. From there the sequence runs in reverse: truncated tetrahedron, tetrahedron, point. The octahedron is the actual shape of the central cut, a three-dimensional piece of a four-dimensional object.

The corners crossed during this passage arrive in groups of 1, 4, 6, 4, 1. This is the Pascal row from Chapter 2 showing up in a moving slice. The first corner starts the film. The next four are the corners of the tetrahedral slice; the middle six are the corners of the octahedron. Each group is a set of corners reached at the same stage of the passage.

To keep track of those encounters, give each corner four coordinates, each either zero or one. The cutting space reaches

corners in order of their coordinate sums. There is one corner with no ones, four with a single one and six with two ones. The remaining groups have three ones and four ones. The middle slice reaches all six corners whose coordinates add to two. For example, (1, 0, 0, 0) belongs to the group of four, while (1, 1, 0, 0) belongs to the middle group of six.

Push a different object, a 5-cell, corner-first through our space and a point grows into a tetrahedron. The tetrahedron reaches the full size of the cell opposite the entering corner, then vanishes. There is no octahedron halfway through this passage.

C. H. Hinton described an early version of such a slice film in *A New Era of Thought* in 1888. He imagined a tesseract moving kata at an inch a minute, with blocks of coloured cubes standing for the successive slices. Ana and kata are his names for the two opposite directions along the fourth axis; kata is the negative direction. His three blocks appeared in succession, each remaining in view for a minute. Chapter 11 explains how his cube system represented these slices.

Turning in a plane

In the animated cube-in-a-cube shadow, the small cube can seem to crawl out through the larger one, expand and enclose it. The tesseract appears to turn itself inside out. Following a single cell through the turn helps explain what happens.

In three dimensions, we usually name a rotation by an axis, the line that stays fixed. The turning takes place in a plane perpendicular to that line. Only in three dimensions does a plane have a single perpendicular direction, making the axis a convenient shortcut. In four dimensions we need to name the plane.

Write a point's position as (x, y, z, w). Each letter measures position along one of four perpendicular directions. Pair those directions and there are six coordinate planes: xy, xz, xw, yz, yw and zw. For comparison, two dimensions give just one coordinate plane,

while three dimensions give three. Adding a fourth direction supplies three new pairings with the existing directions.

An xw-rotation changes x and w while leaving y and z unchanged. Every point with both x and w equal to zero stays where it is. That is an entire fixed plane, the yz-plane.

A simple rotation turns in one plane. A general four-dimensional rotation can turn in two perpendicular planes at once, by separate angles; this is a double rotation. In a double rotation only the centre stays fixed. Chapters 10 and 13 return to the linked circles that arise when the two angles are equal.

Try the turn with the familiar wireframe cube first. Start face-on, with a small back square inside a large front square. Let z be the viewing direction and x a side-to-side direction. Turning in the xz-plane sends the back square sideways. It grows in the perspective image as it comes nearer and becomes a side face. Meanwhile the front square shrinks and becomes the opposite side face. After half a turn, front and back have exchanged places.

Follow a particular face throughout that turn. Its change from small square to sloping side and back to large square belongs to the projected picture. The face remains square, and the cube remains rigid.

Now look at the tesseract along w and turn it in the xw-plane. The small inner cube is the far cell. It begins to slide sideways and swell. At a quarter-turn, 90°, it has become one of the side cells, appearing as a frustum, while a former side cell occupies the inner cube's position. Continue to 180° and the original far cell becomes the near cell. The small inner cube has become the large outer cube.

In the sequence below, follow the marked cell through its changing positions in the image. The y and z coordinates of every corner remain unchanged throughout. The tesseract stays rigid; the changing projection supplies all the apparent stretching.

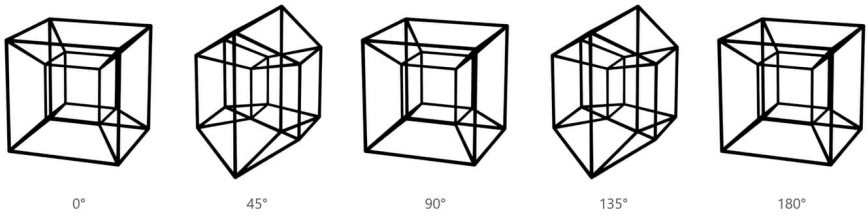


Figure 3.2 Half a turn in the xw-plane, shown at 0°, 45°, 90°, 135° and 180°. The small inner cube becomes the outer one while the tesseract itself stays rigid.

TRY IT Choose the tesseract and set Projection to Persp. Turn on XW only and follow the inner cube through half a turn. Then switch to Stereo and watch the same motion again: straight edges become arcs, but the cells trade places in the same order.

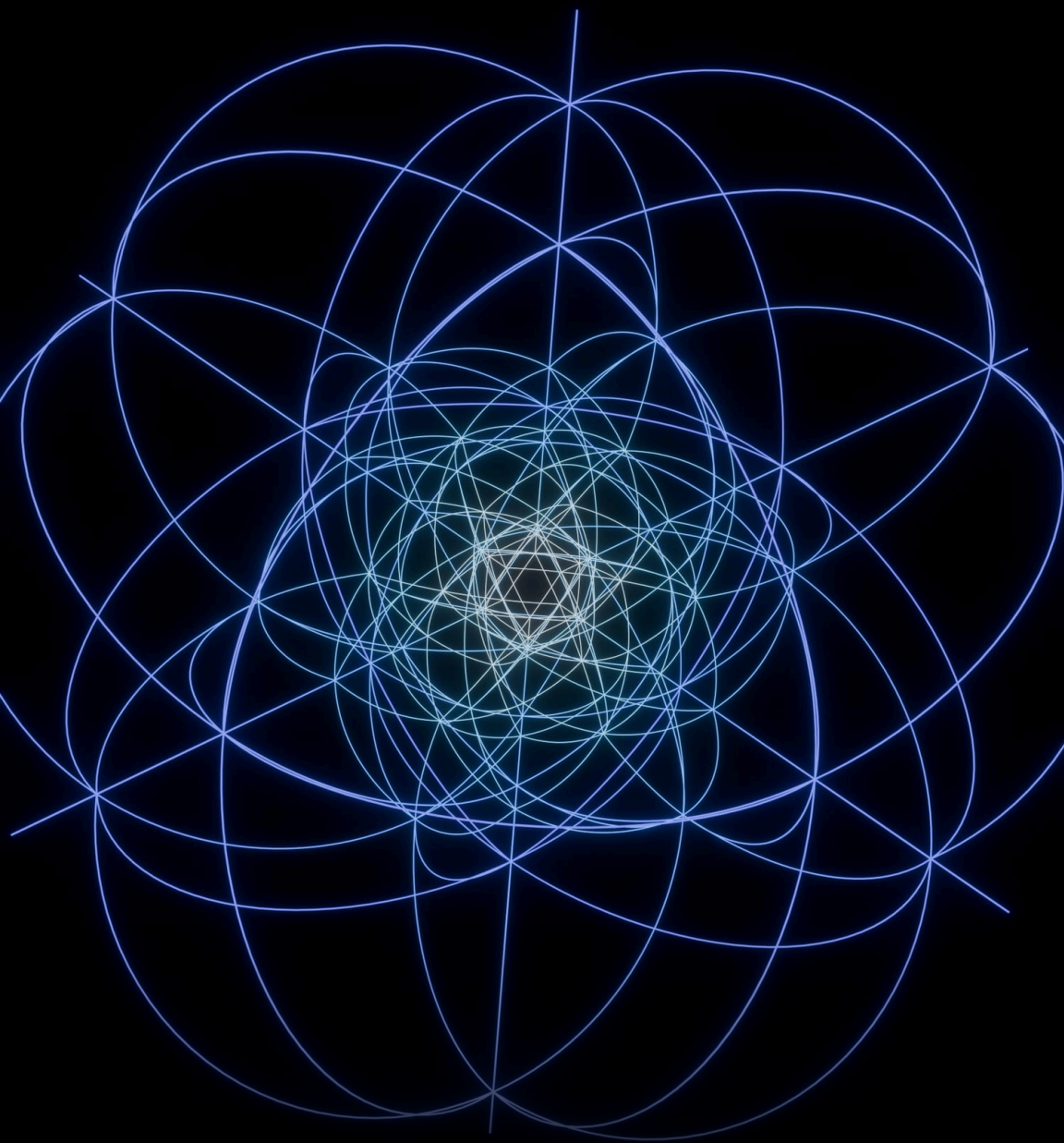
Applied to an object lying in our space, an xw half-turn brings it back with x reversed, making its mirror image: this is the turn Möbius described in 1827, introduced with the glove problem and taken up in Chapter 4.

A. Michael Noll at Bell Telephone Laboratories made some of the earliest computer-animated films of four-dimensional objects. In August 1967 he published his method for displaying higher-dimensional objects in *Communications of the ACM*. He had begun making computer art at Bell Labs in the summer of 1962; prints of his computer-generated animations are in the collection of the Museum of Modern Art in New York.

The page adds another reduction. Every four-dimensional figure here is a flat picture of a three-dimensional projection or cross-section. A projected tesseract on paper is a shadow of a shadow; a pictured slice has also been flattened for the page. Knowing which operation supplied the three-dimensional image helps us read its distortions without assigning them to the original object.

The xw half-turn returns an object to our space as its mirror image through a rigid turn. Chapter 4 follows that turn to the left

hand and the right, and to what Kant thought their difference meant.



The snub 24-cell in stereographic projection. Its cells are 24 icosahedra and 120 tetrahedra.

CHAPTER 4

The Left Hand Problem

Two piles of crystals

Paris, 1848. Jean-Baptiste Biot puts a dish in a corner of his rooms at the Collège de France, where nobody is to touch it. At 74, he has spent years studying substances that twist polarized light. Louis Pasteur, 25, says he can separate a salt that has no effect on this light into two kinds that twist it in opposite directions. Biot is sceptical. He wants to watch.

Biot supplies his own paratartaric acid, also called racemic acid, which he has tested and found to cause no rotation. He brings the soda and ammonia too, and has Pasteur prepare the salt in front of him. About forty-eight hours later, crystals appear. While Biot watches, Pasteur sorts them by their tiny, slanting faces into two piles of mirror-image crystals. Biot makes up the solutions himself. The one that should turn the light left does exactly that.

He takes Pasteur's arm. "My dear boy, I have loved Science so much during my life, that this touches my very heart."

The two forms turn the light by equal amounts in opposite directions; mixed equally, their effects cancel. According to his biographer, Pasteur's first response to the discovery had been "I have it!" Too excited to look through the apparatus again, he rushed out, embraced a curator in the passage and dragged him into the Luxembourg garden to explain.

Pasteur had been lucky with his salt. It forms separate mirror-image crystals only below roughly 28 °C. Above that temperature, it makes a single mixed crystal, leaving nothing for the tweezers to

sort. Scacchi repeated the work in Italy in 1865 above this temperature and obtained only one kind of crystal. Only a small minority of racemic compounds form separate mirror-image crystals at all.

Optical rotation, the twisting of polarized light, was already known. François Arago had seen it in quartz in 1811. In 1820 John Herschel had connected opposite rotations to mirror-image quartz crystals, and Biot had studied rotation in liquids and sugar solutions. Pasteur showed that his salt crystals' mirror-image shapes went with opposite effects in solution.

The difference persisted even after the crystals dissolved. To get a feel for this kind of mismatch, consider two mirror-image snub cubes. These mirror-image solids match in every measurement, yet no turn in our space makes them coincide. A hand and its reflection have the same difficulty.

What Kant noticed

In Königsberg in 1768, Immanuel Kant was about 44, an unsalaried lecturer who also worked as a sublibrarian. His short essay on directions in space gave this mismatch a name: an incongruent counterpart. Two bodies could match in size and shape yet fail to fit within the same boundaries.

Take a rigid copy of your right hand and its exact mirror image. Turn either one as much as you like: the thumb and fingers never all occupy the other's place. Kant also considered screws. A screw winding one way cannot enter a thread winding the other, even when their thickness and the spacing of their turns agree. He found the mismatch in pairs of spherical triangles too, triangles drawn on a sphere.

Kant looked at living things as well, from the swirl of hair at the crown to hops and beans twining opposite ways. He thought almost all snails coiled one way, allowing only about three exceptional species. In the 1929 translation, John Handyside added a footnote: "The facts, as now known, are more complicated than Kant here

suggests". Handyside described about seventeen normally left-coiled mollusc genera, with occasional left-coiled individuals in about two hundred species.

After setting individual exceptions aside, Kant also declared that "all the peoples of the earth are right-handed." Left-handers may feel that the exceptions deserved a little more space in the argument.

A flat cut-out gives us a simpler test. A scalene triangle, one with all its sides different, cannot rotate into its mirror image while staying on the table. Neither can a J-shaped Tetris piece become its L-shaped twin, however far you slide or spin it. Lift it off the table and turn it over, and it fits. The third dimension gives the piece room to flip.

Wittgenstein observed that the problem appears even on a line. Two arrows pointing opposite ways cannot be turned to agree while staying on that line. A half-turn through a plane does the job. As with the flat creatures of Chapter 1, one more direction gives a shape room to reach its mirror image.

Möbius's half-turn

In Leipzig in 1827, the astronomer August Ferdinand Möbius worked through those steps in *Der barycentrische Calcul*. Mirror-image arrangements of points on a line need a half-turn through a plane; mirror-image triangles need a half-turn through three-dimensional space. By analogy, mirror-image solids would need a half-turn through four-dimensional space. Then he dismissed the possibility, writing (in translation): "But since such a space cannot be thought, coincidence in this case is impossible."

The son of a dancing teacher who died when he was three, Möbius had studied astronomy under Gauss in Göttingen in 1813 and held the Leipzig chair of astronomy and higher mechanics from 1816. Much later, in 1858, he and Johann Benedict Listing independently found the one-sided strip that bears Möbius's name. Chapter 13 follows its reversal of handedness; here we need the extra spatial dimension he dismissed in 1827.

As in Chapter 3, call that direction w, alongside our familiar x, y and z. This is a spatial fourth dimension. In a rotation in the xw-plane, x and w change together while y and z stay put. A turn in three dimensions has a fixed line; this four-dimensional turn has a whole fixed plane, the yz-plane.

After 180°, x has become −x and w has become −w. Every point of a solid starting in our space begins with w = 0 and ends there again. Its x-coordinates have all reversed, making it a mirror image across the yz-plane with every internal measurement intact. A point on one side of that plane finishes equally far on the other side. Points already in the plane stay put.

An orthographic projection is a shadow onto our three-dimensional space. At 90°, halfway through the turn, the solid appears flat because its width now extends along w, which the projection leaves out. As the turn continues, the width returns, reversed. The solid stays rigid throughout, and nothing passes through anything else. A cross-section would show a slice through it instead.

TRY IT Choose the tesseract and press Reset View. Hold Q (the XW plane) and follow one cube as it turns: partway round it seems to flatten, then it opens out again on the other side. In four dimensions that cube is rigid the whole time; the flattening belongs to the shadow.

You can make a flexible glove fit the other hand by turning it inside out, as Charles Howard Hinton pointed out. Chapter 11 takes up his work. This trick depends on the glove being hollow and flexible; a solid hand would have to be pulled through itself. The four-dimensional rotation works even with a rigid solid. Wittgenstein wrote: "A right-hand glove could be put on a left hand if it could be turned round in four-dimensional space."

H. G. Wells gave a whole schoolmaster a mirror reversal in "The Plattner Story" in 1896. Gottfried Plattner teaches chemistry despite knowing "nothing beyond the Three Gases (whatever the three gases

may be)." He sets a match to a greenish powder, which explodes, and vanishes for nine days. When he returns, his heart is on the right, his organs are reversed, and he writes right to left with his left hand.

A real condition called situs inversus mirrors the layout of the internal organs, though the molecules remain ordinary. A fully reversed Plattner would have mirror-image molecules as well, raising a question about what he could eat. Wells leaves his schoolmaster's menu unexamined.

Knots fall apart

In the 1860s William Thomson, later Lord Kelvin, proposed that atoms were knotted vortices, swirling loops in a medium called ether. In this model hydrogen was a plain ring, while a trefoil, the simplest true knot, was suggested for carbon. The theory prompted Peter Guthrie Tait to compile knot tables. His 1885 table included knots with up to ten crossings. The atom theory died; the mathematical study of knots continued.

The trefoil comes in left- and right-handed forms. In 1914 Max Dehn proved that they cannot be deformed into each other in three-dimensional space without a strand passing through another. The string here forms a closed loop, with no free end to pull back through. However much you change its outline, its handedness survives.

With access to a spatial fourth dimension, every knotted closed loop can come undone. At a crossing, lift one strand ana, Hinton's word for one direction along the fourth axis. Move it past the other strand and bring it back kata, the opposite direction. The strands pass without meeting because they occupy different positions along the fourth axis.

In the diagram below, switching one crossing lets the trefoil relax into a plain ring. Other knotted loops may need more switches; the extra direction allows each switch without cutting the string.

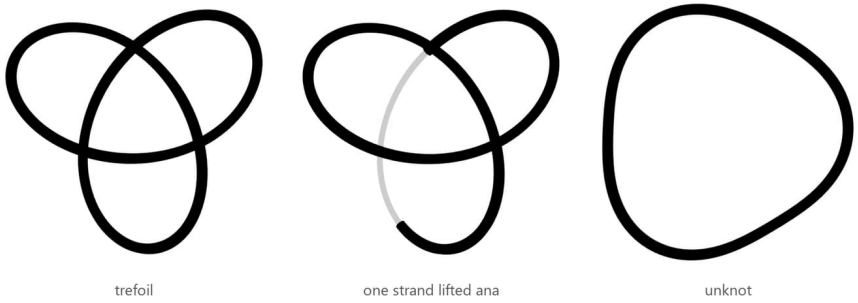


Figure 4.1 One strand lifts ana, passes the other strand and returns, switching a crossing without cutting. A single crossing switch lets the trefoil relax into an unknotted ring.

The surface of a sphere can be knotted in four-dimensional space. The extra freedom releases string but still allows knots in two-dimensional surfaces; Chapter 5 takes up life with this extra room. Chapter 24 examines the 1877 claim that the medium Henry Slade tied sealed-cord knots for the astrophysicist J. C. F. Zöllner, who took them as evidence of a fourth dimension.

Hands all the way down

In 1874 Jacobus Henricus van 't Hoff and, independently, Joseph Achille Le Bel proposed an explanation for optical activity at the molecular scale. Carbon's four bonds point toward the corners of a tetrahedron, a solid with four triangular faces. With four different groups attached, the molecule can be built in two mirror-image ways.

Van 't Hoff first published the idea in a Dutch pamphlet, then in French in 1875 as *La chimie dans l'espace*, or *Chemistry in space*. Hermann Kolbe mocked his spatial chemistry. As it is usually translated, he accused the young man at Utrecht's Veterinary School of preferring "to mount Pegasus (apparently borrowed from the Veterinary School)." Van 't Hoff later received the first Nobel Prize in Chemistry, in 1901, for his work on solutions.

Kelvin coined the word chiral, from the Greek for hand, in a lecture given in 1893 and printed in 1894: "I call any geometrical figure, or group of points, chiral, and say that it has chirality, if its image in a plane mirror, ideally realized, cannot be brought to coincide with itself." Chemists call the two members of a molecular mirror pair enantiomers.

Your nose can tell some enantiomers apart. One form of carvone gives spearmint its minty smell; its mirror image gives caraway its spicy smell. The nose's receptors are themselves handed and respond differently to the two forms, though not every mirror pair smells different. The two forms have matching atoms and connections arranged as mirror images.

Life is homochiral, meaning it uses one hand of particular molecular building blocks. Nineteen of the twenty amino acids used to build proteins are left-handed; glycine has no handedness. The sugars in DNA and RNA are right-handed. Ordinary chemistry produces both hands in equal amounts, and no one knows how life acquired its preference. The suggestion that meteorites supplied the bias remains disputed: some reported excesses have been challenged, while samples from Bennu and Ryugu show no chiral bias.

The ordinary DNA double helix, B-DNA, twists to the right. A left-handed form, Z-DNA, also exists and appears transiently in cells. Andrew Wang, Alexander Rich and colleagues at MIT solved its crystal structure in 1979.

Thalidomide, sold as a mixture of mirror forms, caused severe birth defects in more than 10,000 children. The often-repeated claim that selling only the safe form would have prevented the harm fails because the body converts each form into the other.

Those molecular building blocks bring us back to Plattner's unexamined menu: ordinary food would pose a problem for a fully mirrored man. Lewis Carroll, a mathematics lecturer at Oxford, gave Alice a suitable worry in *Through the Looking-Glass*: "Perhaps Looking-glass milk isn't good to drink".

In December 2024, thirty-eight scientists published a warning in *Science* about mirror life. Bacteria built from mirror-image molecules could evade immune systems and predators that recognize ordinary molecules. The bacteria were hypothetical; the scientists recommended against creating them.

Kant's snails have a modern counterpart in Jeremy, a left-coiled garden snail found in a London compost heap. His reversed body required a left-coiled mate. After a public appeal, the first two lefties sent in mated with each other instead. Jeremy eventually fathered offspring, all right-coiled. A 2020 analysis suggested that left-coiling in these garden snails usually comes from a developmental accident rather than an inherited gene.

Does nature know left from right?

A handed molecule can distinguish another handed molecule. Whether the laws governing them work the same way in a mirror is a separate question.

In summer 1956, Tsung-Dao Lee and Chen-Ning Yang pointed out that no experiment had tested this for the weak nuclear force, the force responsible for beta decay. Chien-Shiung Wu cancelled a planned trip with her husband to run a test with the Low Temperature Group at the US National Bureau of Standards in Washington. They used cobalt-60, whose nuclei emit electrons as they decay.

They cooled cobalt-60 nuclei to about 0.01 K and aligned them in a magnetic field. The electrons emerged more often in one direction relative to the nuclei's spin. In a mirror image of the whole arrangement, that preference reverses, giving a process that the same laws of the weak interaction would not allow. The experiment appeared in print on 15 February 1957.

Wolfgang Pauli had been ready to bet against such a result. On 17 January he wrote: "I do not believe that the Lord is a weak left-hander, and I am ready to bet a very large sum that the experiments will give symmetric results." On the 21st, experimental papers

reached him three hours before a lecture. Afterwards he wrote: "I am shocked not so much by the fact that the Lord prefers the left hand as by the fact that He still appears to be left-right symmetric when He expresses Himself strongly." The strong and electromagnetic forces respect the mirror symmetry that this weak process breaks.

Lee and Yang received the 1957 Nobel Prize in Physics. Wu was not included. In 1978 she received the first Wolf Prize in Physics.

Martin Gardner used the result to answer a communication puzzle, the Ozma problem. Suppose you can talk to aliens by radio but have no shared object to point at. How do you explain which direction you call left? Describing a hand leaves you having to explain which way it faces. Gardner's answer was to describe the cobalt-60 experiment and specify the electrons' direction.

Is handedness a property of space?

> **Speculative.** The philosophical interpretations that follow remain open. The geometric results and Wu's experimental result do not settle which interpretation of space is right.

For his 1768 thought experiment, Kant imagined a single human hand as the first created object. It must be left or right, yet the distances and relations among its parts match those of its mirror counterpart. If space consists only of relations between things, where does the difference reside?

Kant took the mismatch as grounds for believing in absolute space, a space with reality independent of matter. His essay never mentioned a fourth dimension.

He soon abandoned that conclusion. In the *Prolegomena* of 1783, the hand supported his view that space is a form of our perception. He asked: "What can be more similar in every respect and in every part more alike to my hand and to my ear, than their images in a mirror? And yet I cannot put such a hand as is seen in the glass in the

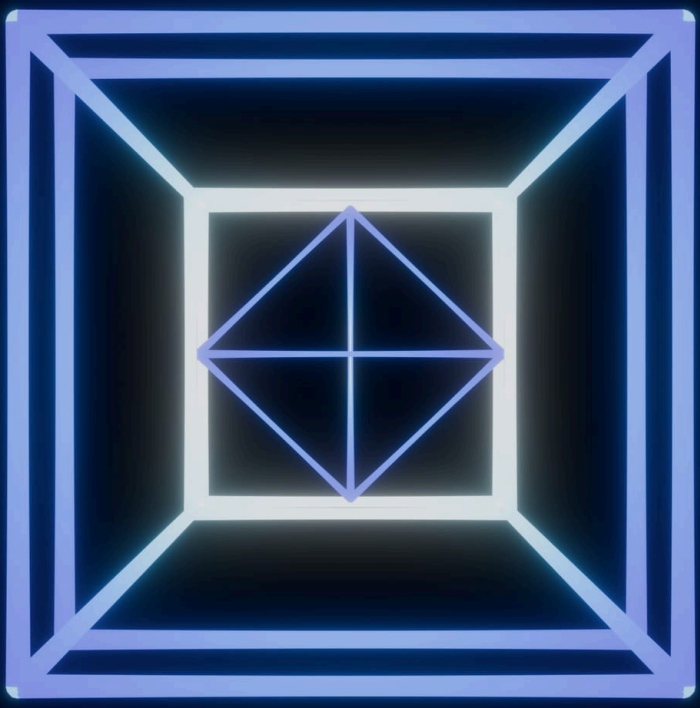
place of its archetype". His answer was: "These objects are not representations of things as they are in themselves, and as the pure understanding would cognise them, but sensuous intuitions, that is, appearances". Chapter 27 returns to this account of how we apprehend space.

In a non-orientable space, one with a Möbius-like twist, a traveller could follow a route and return mirror-reversed without entering a fourth dimension. Whether mirror counterparts can coincide therefore depends on the global shape of their space. Graham Nerlich used this point in 1973 to defend a version of the idea that handedness depends on space itself. Chapter 12 takes up the question of whether our universe has such a shape.

Wu's result suggests another version of Kant's puzzle. Make the first event a cobalt-60 decay, and the laws distinguish its mirror versions. Philosophers still dispute whether that puts handedness in space or in the laws and fields occupying it.

In 1904 Hinton guessed that bilateral symmetry, the matching left and right sides of living bodies, might record a folding in four-dimensional space, like a folded-paper ink blot. His guess remains a historical curiosity with no evidence behind it. Development provides ordinary mechanisms for setting left and right, including beating structures called cilia in an embryo.

Chapter 5 considers what a body in four-dimensional space might be like. A creature there could have four-dimensional hands that no rotation in its space could match to their mirror images. Möbius's half-turn would need room in a fifth dimension.



A tesseract with a small 5-cell placed inside its central cell: a four-dimensional safe and its contents (not to scale).

CHAPTER 5

Life in Four Dimensions

A box of impossible toys

On Friday, 2 June 2017, game designer Marc ten Bosch announced a release on his blog. "Surprise! Today I am releasing something!" He had been developing the four-dimensional puzzle game *Miegakure* since showing a prototype at the Game Developers Conference in 2009. The surprise was *4D Toys*, a box of four-dimensional objects, out that day for iOS and Steam. You could play with a mouse and keyboard, or put on a Vive headset. For players using virtual reality, ten Bosch had some advice: sit on the floor.

Toss a toy between your hands and it behaves much like an ordinary object. Let it bounce off a wall, though, and its visible shape can change. It may disappear altogether. Your hands work in the three-dimensional space you can see; the toy has another direction in which to tumble. You can be sitting on the floor and still lose it from view.

The display shows a cross-section: a three-dimensional slice through a four-dimensional object. A slider moves the slice along the fourth direction, so you can look beyond the space where the toy vanished. The sequence below shows what happens as one tesseract, the four-dimensional counterpart of a cube, passes through our space. Each changing solid is a slice of the same object.



Figure 5.1 A tesseract passing through our three-dimensional space appears, changes shape and vanishes. Each frame shows a slice of the same object.

The box includes six kinds of four-dimensional dice, with 5, 8, 16, 24, 120 and 600 cells. A cell is a three-dimensional part of a four-dimensional object's boundary, equivalent to a face on an ordinary die. The site calls them faces; geometers call them cells. These dice are the six regular polychora, the regular four-dimensional solids examined in Chapters 9 and 10. When one lands on a cell, it rests on a solid piece of its boundary.

If this were your world, what would daily life be like? Start with what you could see, and whether locking the door would do any good.

The extra dimension here is spatial: another direction in space, independent of our familiar three. Chapter 19 takes up time as a fourth dimension. To work out the geometry, we use dimensional analogy: compare life in a plane with life in our space, then carry the comparison up one step. That lets us ask what an extra direction makes possible. An animal that could use it remains speculation.

TRY IT Choose the 5-cell, the simplest polychoron, and turn on XW. Its five tetrahedral cells take turns passing through the middle of the shadow. Every one of them is the same regular tetrahedron; a four-dimensional observer would see no difference between them.

An eye with a solid retina

In Edwin A. Abbott's *Flatland*, published in 1884, A Square has never seen a square. He sees line segments. Abbott demonstrates with a penny on a table: lower your eye to the tabletop and the broad face becomes a line. A Flatlander's eye is always level with its world.

There is no direction within the plane from which to look down at a neighbour's outline.

The Square can still recognize his neighbours. Abbott gives the inhabitants fog, which makes distant parts of a figure look dimmer. Differences in brightness along the visible line let them judge the shape beyond it. They work out a two-dimensional shape from clues in a one-dimensional image, even though they never get our view from above.

Our retinas, the light-receiving surfaces inside our eyes, carry two-dimensional images. We infer three-dimensional depth from clues in those flat pictures. The image has fewer dimensions than the world we see, just as the Square's image does.

An eye built on the same principle in four spatial dimensions would need a three-dimensional retina. Its receptors, the parts that receive light, would fill a solid block. The image itself would have volume, with another direction in which to record the scene. That still leaves a dimension to work out: a four-dimensional observer would infer depth from clues in the three-dimensional image, much as we do from flat pictures. What seeing it would feel like is speculation.

Charles Howard Hinton ran into a practical difficulty in *An Episode of Flatland*: where could a plane creature put two eyes? Its body lacks the thickness needed to place them beside each other as ours are. Before a writer can give a creature depth perception, there is the small matter of finding somewhere to put the eyes.

From a Flatlander's retina to ours to a four-dimensional observer's, the image gains a dimension at each step. A solid array of receptors could hold an image of a whole cube, including its interior.

Look down at a square drawn on paper. You can see its enclosed area because your viewing direction lies outside the plane. By the same reasoning, a four-dimensional observer looking at our three-dimensional space could see every face of a box and its contents at once. None of the walls has any thickness along the extra direction from which the observer looks.

Abbott's Sphere claims this view of Flatland. He can see inside houses and safes, and even into the inhabitants' stomachs. To the Square, a closed boundary hides whatever lies within. From outside the plane, the Sphere can look straight into the enclosed space.

After seeing Flatland's interiors, the Square asks to see the Sphere's own insides. The Sphere cannot oblige. "Would you have me turn my stomach inside out to oblige you?" His advantage over the Square gives him no way to show the interior of his own three-dimensional body.

A four-dimensional observer would face the same limit. Four-dimensional walls could be opaque to it, and another four-dimensional body could hide its interior. This apparent X-ray vision works when looking one dimension down. In its own world, a closed door could still keep things out of sight.

Locks, safes and surgery

The Square has two tablets of accounts in a locked cupboard and holds the key himself. The Sphere sees them anyway, and takes one. "But I descend from Space; the doors, you see, remain unmoved." The Square opens the cupboard to find a tablet missing. Later the visitor reappears in another corner of the room, and the tablet appears on the floor.

The door stayed shut throughout. Its lock blocks an approach within the plane, where entering the cupboard means crossing a boundary line. A hand reaching down from above Flatland can go straight to the enclosed region. From that direction, the closed cupboard is as accessible as an open tray.

Our three-dimensional safes would be just as exposed to an approach along the fourth direction. An imagined four-dimensional hand could reach the contents without crossing a wall, then withdraw the way it came. Ten Bosch uses this access in *Miegakure*, designing levels around feats such as taking objects from locked safes without opening them.

A safe that encloses four-dimensional space needs three-dimensional blocks for walls. The tesseract has eight such walls, its cubic cells, as Chapter 2 develops. In the render below, a small 5-cell sits inside the tesseract's central cell. This is a projection, a flattened view of the whole arrangement, rather than the slice shown by the toys. All eight cubic cells belong to the enclosure.

The Sphere also touches the Square's stomach without breaking through his skin. The Square feels pain inside, although the line enclosing his body remains intact. The Sphere has reached him by the same route that let him take the tablet: from outside the plane, clear of the boundary.

> **Speculative.** Applied to us, that route suggests a four-dimensional surgeon reaching an organ inside a three-dimensional body without making an incision. Geometry supplies a path that avoids the skin. Whether a living hand could follow it, or a patient survive the encounter, belongs to imagined biology. Evidence for such an operation would have to establish a surgeon and what happens to the tissue, beyond showing a possible route. No observation establishes such a surgeon or operation.

Guts, nerves and knots

Eating presents a difficulty in two dimensions. Run a channel through a flat animal from mouth to anus and you cut its body into two pieces. There is no flesh above or below the channel to hold the sides together. Stephen Hawking used this problem in *A Brief History of Time* to question whether two dimensions could support life. It is a constraint on the animal's design, rather than a proof that every possible design must fail.

A. K. Dewdney had already imagined a workaround in his novel *The Planiverse*. His Ardean Yendred has a digestive tract that works like a zipper, opening ahead of the food and sealing behind it. A bite passes through without leaving an open channel across the whole

animal at once. Dewdney keeps the geometric constraint and invents anatomy that gets around it.

Dewdney gives his Ardeans practical solutions for their houses too. Nails are useless for fastening in their world, so they use tape and glue. They build underground, where periodic two-dimensional rivers cannot demolish their homes. Like the zipper gut, these inventions belong to the novel. They show how much everyday engineering a writer has to rethink when a direction is missing.

A three-dimensional animal stays in one piece because material surrounds the tunnel through its body. With a through-gut, we are topologically thick tubes or doughnuts. Topology studies connections while setting aside precise shape: here the question is whether the body stays connected around the hole. In three dimensions, it does.

A four-dimensional body has still more room around a channel. Even removing a two-dimensional sheet from its interior need not disconnect it, just as removing a thread from a three-dimensional solid need not split it apart. The through-gut separates a flat animal; the same problem disappears in three or four dimensions. Whether the imagined animal can digest its dinner remains a biological question.

Flat creatures also have a wiring problem. Two nerves cannot cross in a plane and remain separate. Blood vessels face the same difficulty, which Hawking raised when discussing circulation. Max Tegmark discusses the nerve problem in his work on the number of dimensions.

You can try the same problem with the three-utilities puzzle. Put three houses on a flat sheet and give each a connection to water, gas and electricity. All nine lines must stay separate. However you arrange them, some will cross: the puzzle has no solution on the plane.

In three dimensions, a wire can pass above another wire. Any finite network can be laid out without crossings, even using straight connections between its points. Routes that would have to meet in a

flat layout can now stay separate. That removes the geometric obstruction to wiring an animal, though the actual arrangement of its nerves and blood vessels still needs working out.

Hinton's Astrians have another difficulty. In *An Episode of Flatland*, their eye and both arms sit on one edge of the body. To face the other way, they would have to leave the plane and be flipped. Every Astrian therefore faces east from birth to death. Hinton builds a war around this restriction: attackers arriving from behind can see and strike people who cannot turn to face them.

In four dimensions, string loses its ability to hold a knot. At a crossing, a strand can move into the fourth direction, pass the other strand and return, so every closed loop of one-dimensional string can be unknotted. Chapter 4 gives the full argument.

Surfaces can form knots in four-dimensional space, and Emil Artin constructed examples in 1925. String rings can pass through one another without breaking, a feat that *Miegakure* also uses. A ring that looks securely linked to another in our three-dimensional slice has room to move free along the fourth direction.

How a 4D wheel turns

Hinton gave his higher world machinery. His four-dimensional wheel turns around a plane, so the axle itself has gained a dimension. A flat disc turns around a point at its centre; a wheel in our space turns around a line. Hinton carries the pattern up one more step to an axle plane.

In a simple four-dimensional rotation, a whole plane stays fixed. Four dimensions also allow two planes at right angles to meet at just one point, something two planes in three dimensions cannot do. There is room for rotation in both planes at once. Chapter 3 develops rotation in the xw-plane, containing the ordinary x direction and the fourth direction, w; Chapter 7 supplies the algebra.

Let Hinton's wheel move across our three-dimensional space. First we would see a solid ball completely surrounding a smaller solid ball. The larger is a slice of the wheel, the smaller a slice of its

axle. The larger ball could turn freely around the smaller one. Once the wheel had passed through our space, only the small ball of the axle would remain.

As three-dimensional machinery, this seems to put solid material in the mechanism's way. But we are looking at a slice that leaves out the fourth direction. The visible balls are parts of a larger arrangement, just as a visible toy is only the part of ten Bosch's object that intersects the slice. Hinton's wheel has room to turn beyond the space we see.

The physics problem

Speculative. A four-dimensional animal would have a three-dimensional boundary for skin. Passing through our space, it would appear as a changing solid, then vanish, like one of the toys. Those descriptions follow from geometry. To make the animal live, we would need materials and physical laws that could support it. Carrying our familiar laws into an extra spatial dimension causes trouble even before we get to anatomy.

Light from a lamp would spread over a three-dimensional boundary whose size grows as distance cubed. Double the distance and the same light is spread over eight times as much boundary, making the lamp look one-eighth as bright. Gravity would follow an inverse-cube law too: doubling the distance cuts the force to one-eighth. Hawking's book gives one-fifth for gravity here. That arithmetic is wrong, though his conclusion about unstable orbits stands.

Paul Ehrenfest showed in 1917 that under this extension of the laws, planetary orbits and classical atoms cannot be stable in more than three spatial dimensions. Chapter 21 examines why adding a direction to our physics leaves us without the stable atoms and orbits a habitable world would need.

Waves cause a different difficulty. In two or four spatial dimensions, a sharp pulse leaves a wake: the region behind the advancing wavefront stays disturbed. Signals carried by the waves arrive distorted. Jacques Hadamard observed this failure of Huygens' principle in 1900. The principle describes how a sharp pulse can stay sharp as it spreads; it fails here because the disturbance continues behind the front. An imagined listener might hear a hand-clap as a lingering smear. The wake is established mathematics, while that hearing experience is speculation.

Size changes the design problem too. Four-dimensional bulk grows as length^4, while its three-dimensional boundary grows as length^3. Double every length and you have sixteen times the bulk with eight times the skin. The bulk grows faster than the boundary surrounding it. What that would mean for breathing or strength depends on the imagined animal's materials and organs, so those consequences remain speculative.

> No observation shows a large fourth spatial direction, let alone an animal living in it. Evidence would have to establish that direction and then the biology of its inhabitants. Chapter 22 covers the searches for extra spatial dimensions.

Walking in hyperspace

In *Miegakure*, you can walk through a four-dimensional world while the game takes care of the physics. You see a three-dimensional slice. Press a button and one of its visible dimensions swaps with the fourth. The scene seems to fold and expand like an accordion as the new view appears. A barrier that blocked your path in one slice can give way to a route through another.

The main mechanic rotates the slice. Later you can move it along the extra direction, although then you cannot see where you are going. Ten Bosch read *Flatland* while prototyping, and has connected the game's philosophy to ideas from Japanese gardens. The name *Miegakure* is glossed as in and out of sight; the official subtitle is *Hide and Reveal*.

After the 2009 prototype, *Miegakure* was an Independent Games Festival finalist for design in 2010 and technical excellence in 2011. It won an IndieCade award in 2010, then Best Experimental Game at Sense of Wonder Night in 2014. In January 2026, it remained unreleased, with no release date.

The physics engine built for *Miegakure* reached players through *4D Toys*. Its bouncing objects follow laws of motion generalized to four spatial dimensions. Ten Bosch described the method in "N-Dimensional Rigid Body Dynamics," a paper he wrote alone, accepted to SIGGRAPH 2020 and published in *ACM Transactions on Graphics*. The equations work in any number of dimensions. They let a toy bounce and roll in a simulation without establishing that a habitable universe could work the same way.

There are more than 100 scenes in the toy box. You can play with a ball pit or dominoes, or try fitting a four-dimensional cube through

a hole. It also contains a Klein bottle, a surface that can exist in four dimensions without passing through itself; Chapter 13 examines that construction. Moving the toys and watching their slices change gives you a way to test what an extra direction allows.

CodeParade's *4D Golf* came out on Steam on 22 March 2024, with more than 120 levels across seven worlds. It has a level editor and aids for visualizing four-dimensional space. You can try a route and see where it takes you, or use the editor to build a level of your own. Chapter 27 examines whether practice with games can change how people perceive four dimensions.

Fiction gives visitors bodies that can use the extra direction. In H. G. Wells's "The Plattner Story," a schoolteacher returns from the fourth dimension mirror-reversed, an example of the handedness problem discussed in Chapter 4. In Rudy Rucker's *Spaceland*, Joe Cube grows an eye on a four-dimensional stalk.

For two thousand years, the received answer was that there could be no fourth spatial direction. Chapter 6 turns to the arguments behind that refusal, and how people began to question it.

Keep reading

This sampler ends with Part I. The full book continues through twenty-five more chapters, from Hamilton's quaternions and Schläfli's six polychora to hidden dimensions, time travel, Cubism and the mind.

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